CÁLCULO III
CUADERNO DE EJERCICIOS
SOLUCIONES

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INTRODUCCIÓN

Este documento constituye un material de apoyo para el curso de Cálculo III para las carreras de Economía y Dirección Financiera en el ITAM. Contiene las soluciones detalladas del documento de trabajo *Cálculo III, Cuaderno de Ejercicios*, Lorena Zogaib, Departamento de Matemáticas, ITAM, agosto 1 de 2019.

Todas las soluciones fueron elaboradas por mí, sin una revisión cuidadosa, por lo que seguramente el lector encontrará varios errores en el camino. Ésta es una transcripción en computadora, de mis versiones manuscritas originales. Para este fin, tuve la suerte de contar con la colaboración de dos estudiantes de Economía del ITAM: Angélica Martínez Leyva, que realizó la primera transcripción de las soluciones en Scientific WorkPlace, y Rigel Jarabo García, que elaboró la gran mayoría del material gráfico. Quiero expresar mi agradecimiento a ellas, por el entusiasmo y esmero con que llevaron a cabo su trabajo.

Agradezco de antemano sus comentarios y correcciones en relación con este material.

Lorena Zogaib

CÁLCULO III
TAREA 1 - SOLUCIONES
INTEGRAL INDEFINIDA. INTEGRACIÓN POR SUSTITUCIÓN
(Tema 1.1)

1. (a)

$$\begin{aligned}\int \frac{-3}{\sqrt[5]{x^4}}\,dx &= -3\int x^{-4/5}\,dx \\ &= -3\left(\frac{x^{1/5}}{\frac{1}{5}}\right)+C \\ &= -15x^{1/5}+C.\end{aligned}$$

(b)

$$\begin{aligned}\int \sqrt{\sqrt{\sqrt{x}}}\,dx &= \int \left(\left(x^{1/2}\right)^{1/2}\right)^{1/2}\,dx \\ &= \int x^{1/8}\,dx \\ &= \left(\frac{x^{9/8}}{\frac{9}{8}}\right)+C \\ &= \frac{8}{9}x^{9/8}+C.\end{aligned}$$

(c)

$$\begin{aligned}\int \left(\frac{x}{5}-\frac{2}{x^3}+2\right)\,dx &= \frac{1}{5}\int x\,dx-2\int x^{-3}\,dx+2\int\,dx \\ &= \frac{1}{5}\left(\frac{x^2}{2}\right)-2\left(\frac{x^{-2}}{-2}\right)+2x+C \\ &= \frac{x^2}{10}+\frac{1}{x^2}+2x+C.\end{aligned}$$

(d)

$$\begin{aligned}\int \left(\sqrt{x}+\frac{3}{x}\right)^2\,dx &= \int \left(x+\frac{6}{\sqrt{x}}+\frac{9}{x^2}\right)\,dx \\ &= \int x\,dx+6\int x^{-1/2}dx+9\int x^{-2}\,dx \\ &= \frac{x^2}{2}+6\left(\frac{x^{1/2}}{\frac{1}{2}}\right)+\left(\frac{9x^{-1}}{-1}\right)+C \\ &= \frac{x^2}{2}+12\sqrt{x}-\frac{9}{x}+C\end{aligned}$$

(e)

$$\begin{aligned}
\int \left(\sqrt{e^{2x}} + \ln 2\right)\ dx &= \int \left(\left(e^{2x}\right)^{1/2} + \ln 2\right)\ dx \\
&= \int \left(e^{x} + \ln 2\right)\ dx \\
&= e^{x} + (\ln 2)\, x + C \\
&= e^{x} + x \ln 2 + C.
\end{aligned}$$

2. (a) $f'(x) = 7x + \cos x$.

$$\begin{aligned}
f(x) &= \int f'(x)\ dx \\
&= \int (7x + \cos x)\ dx \\
&= \frac{7x^2}{2} + \operatorname{sen} x + C.
\end{aligned}$$

(b) $f''(x) = 9x^2 + 6x$.

$$\begin{aligned}
f'(x) &= \int f''(x)\ dx \\
&= \int \left(9x^2 + 6x\right)\ dx \\
&= 3x^3 + 3x^2 + C_1,
\end{aligned}$$

$$\begin{aligned}
f(x) &= \int f'(x)\, dx \\
&= \int \left(3x^3 + 3x^2 + C_1\right) dx \\
&= \frac{3}{4}x^4 + x^3 + C_1 x + C_2.
\end{aligned}$$

(c) $f''(x) = -1$.

$$\begin{aligned}
f'(x) &= \int f''(x)\ dx \\
&= \int (-1)\ dx \\
&= -x + C_1,
\end{aligned}$$

$$\begin{aligned}
f(x) &= \int f'(x)\ dx \\
&= \int (-x + C_1)\ dx \\
&= -\frac{x^2}{2} + C_1 x + C_2.
\end{aligned}$$

(d) $f''(x) = 0$.

$$\begin{aligned} f'(x) &= \int f''(x)\ dx \\ &= \int 0\ dx = C_1, \end{aligned}$$

$$\begin{aligned} f(x) &= \int f'(x)\, dx \\ &= \int C_1\ dx \\ &= C_1 x + C_2. \end{aligned}$$

3. (a) $\dfrac{dy}{dx} = \dfrac{1}{x^2} + x,\ \ y(2) = 1.$

Se tiene

$$y(x) = \int \left(\frac{dy}{dx}\right)\ dx = \int \left(\frac{1}{x^2} + x\right)\ dx = -\frac{1}{x} + \frac{x^2}{2} + C.$$

Sabemos que

$$y(2) = 1 = -\frac{1}{2} + \frac{2^2}{2} + C \qquad \therefore \qquad C = -\frac{1}{2}.$$

Por lo tanto,

$$y(x) = -\frac{1}{x} + \frac{x^2}{2} - \frac{1}{2}.$$

(b) $\dfrac{d^2 s}{dt^2} = \dfrac{3t}{8},\ \ s(4) = 4,\ \ s'(4) = 3.$

Se tiene

$$\begin{aligned} s'(t) &= \int s''(t)\ dt = \int \frac{3t}{8}\ dt = \frac{3}{16}t^2 + C_1, \\ s(t) &= \int s'(t)\ dt = \int \left(\frac{3}{16}t^2 + C_1\right)\ dt = \frac{t^3}{16} + C_1 t + C_2. \end{aligned}$$

Sabemos que

$$\begin{aligned} s(4) &= 4 = \frac{4^3}{16} + 4C_1 + C_2 \\ s'(4) &= 3 = \frac{3}{16}4^2 + C_1 \\ &\therefore \quad C_1 = 0, \quad C_2 = 0. \end{aligned}$$

Por lo tanto,

$$s(t) = \frac{t^3}{16}.$$

4. Sabemos que la utilidad marginal es $u'(q) = 100 - 2q$ y que $u(10) = 700$. Para maximizar la utilidad $u(q)$ es necesario que $u'(q) = 0$. Entonces,

$$u'(q) = 0 = 100 - 2q \quad \therefore \quad q = 50.$$

Observamos que se trata de un problema de maximización, ya que

$$u''(q) = -2 < 0.$$

Para encontrar la utilidad máxima $u_{máx}$ debemos encontrar la función $u(q)$. Se tiene

$$u(q) = \int u'(q)\ dq = \int (100 - 2q)\ dq = 100q - q^2 + A.$$

Sabemos que

$$u(10) = 700 = 100(10) - (10)^2 + A \quad \therefore \quad A = -200.$$

Así, la función de utilidad es

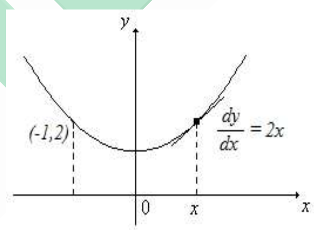
$$u(q) = 100q - q^2 - 200.$$

Concluimos que la empresa maximiza su utilidad en $q = 50$, con

$$u_{máx} = u(50) = 2,300 \text{ dólares.}$$

5. En cada punto (x, y) de la curva $y(x)$ la pendiente es el doble de la abscisa, esto es,

$$\frac{dy}{dx} = 2x.$$



De esta manera,

$$y(x) = \int \left(\frac{dy}{dx}\right)\ dx = \int 2x\ dx = x^2 + C.$$

La curva pasa por el punto $(-1, 2)$, de donde

$$y(-1) = 2 = (-1)^2 + C \quad \therefore \quad C = 1.$$

Por lo tanto, la curva que buscamos es

$$y(x) = x^2 + 1.$$

6. (a) $\int \sqrt{1-2x}\, dx$.

Sea
$u = 1 - 2x$
$du = -2dx$.
Así,

$$\begin{aligned} \int \sqrt{1-2x}\, dx &= -\frac{1}{2}\int \sqrt{1-2x}\ (-2dx) \\ &= -\frac{1}{2}\int \sqrt{u}\, du \\ &= -\frac{1}{3}u^{3/2} + C \\ &= -\frac{1}{3}(1-2x)^{3/2} + C. \end{aligned}$$

(b) $\int \frac{y-1}{\sqrt{y^2-2y+1}}\, dy$.

Sea
$u = y^2 - 2y + 1$
$du = (2y-2)\, dy = 2(y-1)\, dy$.
Así,

$$\begin{aligned} \int \frac{y-1}{\sqrt{y^2-2y+1}}\, dy &= \frac{1}{2}\int \frac{2(y-1)}{\sqrt{y^2-2y+1}}\, dy \\ &= \frac{1}{2}\int \frac{du}{\sqrt{u}} \\ &= \sqrt{u} + C \\ &= \sqrt{y^2-2y+1} + C \\ &= \sqrt{(y-1)^2} + C \\ &= |y-1| + C. \end{aligned}$$

(c) $\int \frac{7r^3}{\sqrt[5]{1-r^4}}\, dr$.

Sea
$u = 1 - r^4$
$du = -4r^3 dr$
Así,

$$\begin{aligned} \int \frac{7r^3}{\sqrt[5]{1-r^4}}\, dr &= -\frac{7}{4}\int \frac{-4r^3}{\sqrt[5]{1-r^4}}\, dr \\ &= -\frac{7}{4}\int u^{-1/5}\, du \\ &= -\frac{35}{16}u^{4/5} + C \\ &= -\frac{35}{16}\left(1-r^4\right)^{4/5} + C. \end{aligned}$$

(d) $\int \frac{3x}{(3x^2+3)^3}\ dx.$

Sea

$u = 3x^2 + 3$

$du = 6x\ dx$

Así,

$$\begin{aligned}
\int \frac{3x}{(3x^2+3)^3}\ dx &= \frac{1}{2}\int \frac{(6x\ dx)}{(3x^2+3)^3} \\
&= \frac{1}{2}\int \frac{du}{u^3} \\
&= \frac{1}{2}\int u^{-3}\ du \\
&= -\frac{1}{4}u^{-2} + C \\
&= -\frac{1}{4(3x^2+3)^2} + C.
\end{aligned}$$

(e) $\int sen(-4x)\ \ dx.$

Sea

$u = -4x$

$du = -4\ dx.$

Así,

$$\begin{aligned}
\int \operatorname{sen}(-4x)\ \ dx &= -\frac{1}{4}\int \operatorname{sen}(-4x)\,(-4\ dx) \\
&= -\frac{1}{4}\int \operatorname{sen} u\ du \\
&= \frac{1}{4}\cos u + C \\
&= \frac{1}{4}\cos(-4x) + C.
\end{aligned}$$

(f) $\int sen^5(x/3)\cos(x/3)\ \ dx.$

Sea

$u = \operatorname{sen}(x/3)$

$du = \frac{1}{3}\cos(x/3)\ \ dx.$

Así,

$$\begin{aligned}
\int \operatorname{sen}^5(x/3)\cos(x/3)\ \ dx &= 3\int \operatorname{sen}^5(x/3)\left[\frac{1}{3}\cos(x/3)\ \ dx\right] \\
&= 3\int u^5\ du \\
&= \frac{u^6}{2} + C \\
&= \frac{1}{2}\operatorname{sen}^6(x/3) + C.
\end{aligned}$$

(g) $\int 2x^2 \cos\left(x^3\right)\ dx.$

Sea
$$u = x^3$$
$$du = 3x^2\ dx.$$
Así,

$$\begin{aligned}
\int 2x^2 \cos\left(x^3\right)\ dx &= \frac{2}{3}\int \cos\left(x^3\right)\left(3x^2\ dx\right) \\
&= \frac{2}{3}\int \cos u\ du \\
&= \frac{2}{3}\operatorname{sen} u + C \\
&= \frac{2}{3}\operatorname{sen}\left(x^3\right) + C.
\end{aligned}$$

(h) $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)\ dx.$

Sea
$$u = \frac{1}{x}$$
$$du = -\frac{1}{x^2}\ dx.$$
Así,

$$\begin{aligned}
\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)\ dx &= -\int \cos^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\ dx\right) \\
&= -\int \cos^2 u\ du \\
&= -\left[\frac{u}{2} + \frac{\operatorname{sen}(2u)}{4}\right] + C \\
&= -\frac{1}{2x} - \frac{1}{4}\operatorname{sen}\left(\frac{2}{x}\right) + C.
\end{aligned}$$

(i) $\int \cos x\sqrt{1 - 2\operatorname{sen} x}\ dx.$

Sea
$$u = 1 - 2\operatorname{sen} x$$
$$du = -2\cos x\ dx.$$
Así,

$$\begin{aligned}
\int \cos x\sqrt{1 - 2\operatorname{sen} x}\ dx &= -\frac{1}{2}\int \sqrt{1 - 2\operatorname{sen} x}(-2\cos x)\ dx \\
&= -\frac{1}{2}\int \sqrt{u}\ du \\
&= -\frac{1}{3}u^{3/2} + C \\
&= -\frac{1}{3}\left(1 - 2\operatorname{sen} x\right)^{3/2} + C.
\end{aligned}$$

(j) $\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx.$

Sea

$$u = 1+\sqrt{x}$$
$$du = \frac{1}{2\sqrt{x}}\,dx.$$

Así,

$$\begin{aligned}
\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx &= 2\int \frac{\left(\frac{1}{2\sqrt{x}}\,dx\right)}{(1+\sqrt{x})^2} \\
&= 2\int \frac{du}{u^2} \\
&= -\frac{2}{u}+C \\
&= -\frac{2}{1+\sqrt{x}}+C.
\end{aligned}$$

(k) $\int x\sqrt{3x+2}\,dx.$

Sea

$$u = 3x+2$$
$$du = 3\,dx$$
$$x = \frac{u-2}{3}.$$

Así,

$$\begin{aligned}
\int x\sqrt{3x+2}\,dx &= \frac{1}{3}\int x\sqrt{3x+2}\,(3\,dx) \\
&= \frac{1}{3}\int \left(\frac{u-2}{3}\right)\sqrt{u}\,du \\
&= \frac{1}{9}\int \left(u^{3/2}-2u^{1/2}\right)\,du \\
&= \frac{2}{45}u^{5/2}-\frac{4}{27}u^{3/2}+C \\
&= \frac{2}{45}(3x+2)^{5/2}-\frac{4}{27}(3x+2)^{3/2}+C.
\end{aligned}$$

(l) $\int \frac{e^{2x}}{e^2+e^{2x}}\,dx.$

Sea

$$u = e^2+e^{2x}$$
$$du = \left(0+2e^{2x}\right)\,dx = 2e^{2x}\,dx$$

Así,

$$\begin{aligned}
\int \frac{e^{2x}}{e^2+e^{2x}}\,dx &= \frac{1}{2}\int \frac{2e^{2x}\,dx}{e^2+e^{2x}} \\
&= \frac{1}{2}\int \frac{du}{u} \\
&= \frac{1}{2}\ln|u|+C \\
&= \frac{1}{2}\ln\left|e^2+e^{2x}\right|+C \\
&= \frac{1}{2}\ln(e^2+e^{2x})+C.
\end{aligned}$$

(m) $\displaystyle\int \frac{1}{e^x}\sqrt{1+\frac{1}{e^x}}\,dx = \int e^{-x}\sqrt{1+e^{-x}}\,dx.$

Sea

$u = 1+e^{-x}$

$du = -e^{-x}\,dx$

Así,

$$\begin{aligned}
\int \frac{1}{e^x}\sqrt{1+\frac{1}{e^x}}\,dx &= \int e^{-x}\sqrt{1+e^{-x}}\,dx \\
&= -\int \sqrt{1+e^{-x}}\left(-e^{-x}\right)dx \\
&= -\int \sqrt{u}\,du \\
&= -\frac{2}{3}u^{3/2}+C \\
&= -\frac{2}{3}\left(1+e^{-x}\right)^{3/2}+C.
\end{aligned}$$

(n) $\displaystyle\int \frac{2^{1/x}}{x^2}\,dx.$

Sea

$u = \dfrac{1}{x}$

$du = -\dfrac{1}{x^2}\,dx$

Así,

$$\begin{aligned}
\int \frac{2^{1/x}}{x^2}\,dx &= -\int 2^{1/x}\left(-\frac{1}{x^2}\right)dx \\
&= -\int 2^u\,du \\
&= -\frac{1}{\ln 2}\,2^u+C \\
&= -\frac{1}{\ln 2}\,2^{1/x}+C.
\end{aligned}$$

(o) $\displaystyle\int \frac{\ln\left(4\sqrt{x}\right)}{x}\,dx.$

Sea

$$u = \ln\left(4\sqrt{x}\right) = \ln 4 + \tfrac{1}{2}\ln x$$
$$du = \frac{1}{2x}\,dx$$

Así,

$$\begin{aligned}
\int \frac{\ln\left(4\sqrt{x}\right)}{x}\,dx &= \int 2\ln\left(4\sqrt{x}\right)\left(\frac{1}{2x}\right)dx \\
&= \int 2u\,du \\
&= u^2 + C \\
&= \ln^2\left(4\sqrt{x}\right) + C.
\end{aligned}$$

(p) $\int \left(e^x + e^{-x}\right)^2\,dx = \int \left[\left(e^x\right)^2 + 2\left(e^x\right)\left(e^{-x}\right) + \left(e^{-x}\right)^2\right]\,dx = \int \left[e^{2x} + 2 + e^{-2x}\right]\,dx$
$= \int e^{2x}\,dx + \int 2\,dx + \int e^{-2x}\,dx.$

Sean

$$u = 2x \qquad \text{y} \qquad w = -2x$$
$$du = 2\,dx \qquad\qquad dw = -2\,dx.$$

Así,

$$\begin{aligned}
\int \left(e^x + e^{-x}\right)^2 dx &= \frac{1}{2}\int e^{2x}\left(2\,dx\right) + \int 2\,dx - \frac{1}{2}\int e^{-2x}\left(-2\,dx\right) \\
&= \frac{1}{2}\int e^u\,du + \int 2\,dx - \frac{1}{2}\int e^w\,dw = \frac{1}{2}e^u + 2x - \frac{1}{2}e^w + C \\
&= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C.
\end{aligned}$$

(q) $\int \left(3^x + 1\right)^2\,dx = \int \left[\left(3^x\right)^2 + 2\left(3^x\right)(1) + 1\right]\,dx = \int \left[3^{2x} + 2\left(3^x\right) + 1\right]\,dx.$

Sea

$$u = 2x$$
$$du = 2\,dx.$$

Así,

$$\begin{aligned}
\int \left(3^x + 1\right)^2 dx &= \frac{1}{2}\int 3^{2x}\left(2\,dx\right) + 2\int 3^x\,dx + \int\,dx \\
&= \frac{1}{2}\int 3^u\,du + 2\int 3^x\,dx + \int\,dx \\
&= \frac{1}{2}\frac{1}{\ln 3}3^u + 2\frac{1}{\ln 3}3^x + x + C \\
&= \frac{1}{2\ln 3}3^{2x} + \frac{2}{\ln 3}3^x + x + C.
\end{aligned}$$

(r) $\int x^{3\alpha}\, d\alpha$.

Sea
$u = 3\alpha$
$du = 3\, d\alpha$.
Así,

$$\begin{aligned}
\int x^{3\alpha}\, d\alpha &= \frac{1}{3}\int x^{3\alpha}\,(3\, d\alpha) \\
&= \frac{1}{3}\int x^{u}\, du \\
&= \frac{1}{3}\,\frac{1}{\ln x}\, x^{u} + C \\
&= \frac{1}{3\ln x}\, x^{3\alpha} + C.
\end{aligned}$$

7. (a) Sean $a \neq 0$, $p \neq -1$. Sea
$u = ax + b$
$du = a\, dx$.
Así,

$$\begin{aligned}
\int (ax+b)^{p}\, dx &= \frac{1}{a}\int (ax+b)^{p}\,(a\, dx) \\
&= \frac{1}{a}\int u^{p}\, du \\
&= \frac{1}{a}\left(\frac{u^{p+1}}{p+1}\right) + C \\
&= \frac{1}{a(p+1)}\,(ax+b)^{p+1} + C.
\end{aligned}$$

(b) i) $\int (2x+1)^{4}\, dx$.

Identificamos $a = 2$, $b = 1$ y $p = 4$. Así,

$$\begin{aligned}
\int (2x+1)^{4}\, dx &= \frac{1}{2(4+1)}\,(2x+1)^{4+1} + C \\
&= \frac{1}{10}\,(2x+1)^{5} + C.
\end{aligned}$$

ii) $\int \frac{1}{\sqrt{4-x}}\, dx$.

Identificamos $a = -1$, $b = 4$ y $p = -1/2$. Así,

$$\begin{aligned}
\int \frac{1}{\sqrt{4-x}}\, dx &= \frac{1}{(-1)(-\frac{1}{2}+1)}\,(4-x)^{(-1/2)+1} + C \\
&= -2\sqrt{4-x} + C.
\end{aligned}$$

CÁLCULO III
TAREA 2 - SOLUCIONES
SUMAS FINITAS. SUMAS DE RIEMANN. INTEGRAL DEFINIDA.
(Tema 1.2)

1. (a) $\displaystyle\sum_{k=1}^{100}(-1)^k 2^{k-1} = -1 + 2 - 2^2 + \cdots + 2^{99}.$

   (b) $\displaystyle\sum_{i=0}^{n}(-1)^i \cos(ix) = 1 - \cos(x) + \cos(2x) - \cos(3x) + \cdots + (-1)^n \cos(nx).$

   (c) $\displaystyle\sum_{n=j}^{j+2} n^2 = j^2 + (j+1)^2 + (j+2)^2.$

   (d) $\displaystyle\sum_{k=1}^{n} f(x_k)\,\Delta x_k = f(x_1)\,\Delta x_1 + f(x_2)\,\Delta x_2 + \cdots + f(x_n)\,\Delta x_n.$

2. (a) $\displaystyle\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{50} = \sum_{k=1}^{25} \frac{1}{2k}.$

   (b) $\displaystyle 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots - \frac{1}{15} = \sum_{k=1}^{8} (-1)^{k+1} \frac{1}{2k-1}.$

   (c) $\displaystyle -\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5} = \frac{1}{5}(-1 + 2 - 3 + 4 - 5) = \frac{1}{5}\sum_{k=1}^{5} (-1)^k\ k = -\frac{1}{5}\sum_{k=1}^{5}(-1)^{k+1}\ k.$

   (d) $\displaystyle x^{-2} + x^{-1} + x^0 + x^1 = \sum_{k=1}^{4} x^{k-3}.$

3. i) $\displaystyle\sqrt{3} - \sqrt{4} + \sqrt{5} - \sqrt{6} = \sum_{k=1}^{4} (-1)^{k+1}\sqrt{k+2}.$

   ii) $\displaystyle\sqrt{3} - \sqrt{4} + \sqrt{5} - \sqrt{6} = \sum_{k=3}^{6} (-1)^{k+1}\sqrt{k}.$

   iii) $\displaystyle\sqrt{3} - \sqrt{4} + \sqrt{5} - \sqrt{6} = \sum_{k=-5}^{-2} (-1)^{k+1}\sqrt{k+8}.$

4. (a) $\displaystyle\sum_{k=1}^{n} 2k(1+3k) = 2\sum_{k=1}^{n} k + 6\sum_{k=1}^{n} k^2 = \frac{2n(n+1)}{2} + \frac{6n(n+1)(2n+1)}{6} = 2n(n+1)^2.$

(b)

$$\begin{aligned}\sum_{k=3}^{99}\left(\frac{1}{k}-\frac{1}{k+1}\right) &= (\frac{1}{3}\ \underbrace{-\frac{1}{4})+(\frac{1}{4}}\ \underbrace{-\frac{1}{5})+(\frac{1}{5}}\ -\cdots\underbrace{-\frac{1}{98})+(\frac{1}{98}}\ \underbrace{-\frac{1}{99})+(\frac{1}{99}}\ -\frac{1}{100}) \\ &= \frac{1}{3}-\frac{1}{100}=\frac{97}{300}.\end{aligned}$$

(c)

$$\begin{aligned}\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j) &= \sum_{i=1}^{2}\left[(i+1)+(i+2)+(i+3)\right] \\ &= \sum_{i=1}^{2}(3i+6) \\ &= [3\,(1)+6]+[3\,(2)+6]=21.\end{aligned}$$

5.

$$\sum_{i=1}^{n} i=\frac{n\,(n+1)}{2}=78$$

$$\begin{aligned}&\therefore\ \ n\,(n+1)=156 \\ &\therefore\ \ n^2+n-156=0 \\ &\therefore\ \ (n+13)\,(n-12)=0 \\ &\therefore\ \ n=12 \qquad (\text{se descarta } n=-13).\end{aligned}$$

6. Sean $x_1=3$, $x_2=4$, $x_3=1$.

(a) $\mu=\dfrac{1}{3}\displaystyle\sum_{k=1}^{3}x_i=\dfrac{1}{3}\,(x_1+x_2+x_3)=\dfrac{1}{3}\,(3+4+1)=\dfrac{8}{3}.$

(b) $$\begin{aligned}\sigma^2=\frac{1}{3}\sum_{k=1}^{3}x_i^2-\left(\frac{1}{3}\sum_{k=1}^{3}x_i\right)^2 &= \frac{1}{3}\,(x_1^2+x_2^2+x_3^2)-\left[\frac{1}{3}\,(x_1+x_2+x_3)\right]^2 \\ &= \frac{1}{3}\,(3^2+4^2+1^2)-\left[\frac{1}{3}\,(3+4+1)\right]^2=\frac{14}{9}.\end{aligned}$$

7. Sean $r\neq 1$ y $n\geq 1$.

(a) Sea

$$S_n=\sum_{k=0}^{n-1}r^k=1+r+r^2+r^3+\cdots+r^{n-2}+r^{n-1}.$$

Multiplicando por r ambos lados de la igualdad, se obtiene

$$rS_n=r+r^2+r^3+r^4+\cdots+r^{n-1}+r^n,$$

de donde

$$\begin{aligned} S_n - rS_n &= 1 - r^n \\ (1-r)S_n &= 1 - r^n \\ S_n &= \frac{1-r^n}{1-r}. \end{aligned}$$

$$\therefore \sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}.$$

(b) Partimos de la suma geométrica ($r \neq 1$, $k \geq 1$)

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}.$$

Esto es,

$$1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}.$$

Derivando con respecto a r ambos lados de la igualdad, se tiene

$$\frac{d}{dr}\left(1 + r + r^2 + r^3 + \cdots + r^{n-1}\right) = \frac{d}{dr}\left(\frac{1-r^n}{1-r}\right)$$

$$\therefore\ 1 + 2r + 3r^2 + \cdots + (n-1)\, r^{n-2} = \frac{1 - nr^{n-1} + (n-1)r^n}{(1-r)^2}$$

Multiplicando por r ambos lados de la igualdad se llega a

$$r + 2r^2 + 3r^3 + \cdots + (n-1)\, r^{n-1} = \frac{r\left[1 - nr^{n-1} + (n-1)r^n\right]}{(1-r)^2}$$

$$\therefore\ \sum_{k=0}^{n-1} kr^k = \frac{r\left[1 - nr^{n-1} + (n-1)r^n\right]}{(1-r)^2}.$$

(c) De los incisos 7a y 7b se sigue

$$\sum_{k=0}^{n-1}(a+kd)r^k = a\sum_{k=0}^{n-1} r^k + d\sum_{k=0}^{n-1} kr^k = \frac{a(1-r^n)}{1-r} + \frac{rd\left[1 - nr^{n-1} + (n-1)r^n\right]}{(1-r)^2}.$$

8. En los primeros incisos se trata de una suma geométrica. El último inciso corresponde a una suma aritmético-geométrica.

(a)

$$\begin{aligned} \sum_{k=0}^{99}(-1)^k\left(\frac{1}{2}\right)^k &= \sum_{k=0}^{99}\left(-\frac{1}{2}\right)^k = \frac{1-\left(-\frac{1}{2}\right)^{100}}{1-\left(-\frac{1}{2}\right)} \\ &= \frac{2}{3}\left[1-\left(-\frac{1}{2}\right)^{100}\right] = \frac{2}{3}\left[1-\left(\frac{1}{2}\right)^{100}\right]. \end{aligned}$$

(b) Esta es la misma suma que la del inciso anterior, lo que se demuestra haciendo el cambio de índices $m = k - 1$:

$$\sum_{k=1}^{100} (-1)^{k-1} \left(\frac{1}{2}\right)^{k-1} = \sum_{m=0}^{99} (-1)^m \left(\frac{1}{2}\right)^m = \sum_{k=0}^{99} (-1)^k \left(\frac{1}{2}\right)^k.$$

(c)

$$\begin{aligned}
\sum_{k=1}^{40} \left[\frac{2^k}{(-3)^k} + 5\right] &= \sum_{k=1}^{40} \left(-\frac{2}{3}\right)^k + \sum_{k=1}^{40} 5 = \left(-\frac{2}{3}\right) \sum_{k=1}^{40} \left(-\frac{2}{3}\right)^{k-1} + \sum_{k=1}^{40} 5 \\
&= \left(-\frac{2}{3}\right) \frac{1 - (-\frac{2}{3})^{40}}{1 - (-\frac{2}{3})} + (5)(40) = \left(-\frac{2}{5}\right) \left[1 - \left(-\frac{2}{3}\right)^{40}\right] + 200 \\
&= \left(-\frac{2}{5}\right) \left[1 - \left(\frac{2}{3}\right)^{40}\right] + 200.
\end{aligned}$$

(d)

$$\begin{aligned}
\sum_{k=1}^{40} \frac{2^k}{(-3)^k} + 5 &= \left[\left(-\frac{2}{3}\right) \sum_{k=1}^{40} \left(-\frac{2}{3}\right)^{k-1}\right] + 5 \\
&= \left(-\frac{2}{3}\right) \frac{1 - (-\frac{2}{3})^{40}}{1 - (-\frac{2}{3})} + 5 = \left(-\frac{2}{5}\right) \left[1 - \left(-\frac{2}{3}\right)^{40}\right] + 5 \\
&= \left(-\frac{2}{5}\right) \left[1 - \left(\frac{2}{3}\right)^{40}\right] + 5.
\end{aligned}$$

(e) Usando el resultado del ejercicio 7b, se obtiene

$$\begin{aligned}
\sum_{k=0}^{99} \frac{k}{2^k} &= \sum_{k=0}^{99} k \left(\frac{1}{2}\right)^k = \frac{\left(\frac{1}{2}\right) \left[1 - 100 \left(\frac{1}{2}\right)^{99} + 99 \left(\frac{1}{2}\right)^{100}\right]}{\left(1 - \left(\frac{1}{2}\right)\right)^2} \\
&= 2 \left[1 - 100 \left(\frac{1}{2}\right)^{99} + 99 \left(\frac{1}{2}\right)^{100}\right].
\end{aligned}$$

9. (a) $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} (2c_k^2 - 5c_k)\, \Delta x_k = \int_0^1 (2x^2 - 5x)\, dx.$

(b) $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^2}\, \Delta x_k = \int_0^1 \sqrt{4 - x^2} dx.$

(c) $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} \cos(c_k)\, \Delta x_k = \int_{\pi/2}^{\pi} \cos x \; dx.$

10. Hay que identificar $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} f(c_k)\Delta x = \int\limits_a^b f(x)\,dx$, donde $c_k = a + k\Delta x$ y $\Delta x = \dfrac{b-a}{n}$.

(a) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} \underbrace{\left(\frac{3k}{n}\right)^2}_{f(c_k)} \underbrace{\left(\frac{3}{n}\right)}_{\Delta x}$.

Sea $c_k = \dfrac{3k}{n}$. Se tiene

$$c_k = \frac{3k}{n} \quad\Longrightarrow\quad f(c_k) = c_k^2.$$

Como

$$c_k = \frac{3k}{n} = 0 + k\left(\frac{3}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{3}{n} = \frac{b-a}{n}, \\ b &= 3 + a = 3 + 0 = 3. \end{aligned}$$

Así,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{3k}{n}\right)^2\left(\frac{3}{n}\right) = \int\limits_0^3 x^2\,dx.$$

(b) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} \underbrace{\left(2+\frac{4k}{n}\right)^2}_{f(c_k)} \underbrace{\left(\frac{4}{n}\right)}_{\Delta x}$.

Aquí hay dos respuestas directas:

(b.1) Sea $c_k = 2 + \dfrac{4k}{n}$. Se tiene

$$c_k = 2 + \frac{4k}{n} \quad\Longrightarrow\quad f(c_k) = c_k^2.$$

Como

$$c_k = 2 + \frac{4k}{n} = 2 + k\left(\frac{4}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 2, \\ \Delta x &= \frac{4}{n} = \frac{b-a}{n}, \\ b &= 4 + a = 4 + 2 = 6. \end{aligned}$$

Así,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(2+\frac{4k}{n}\right)^2\left(\frac{4}{n}\right) = \int\limits_2^6 x^2\,dx.$$

(b.2) Sea $c_k = \dfrac{4k}{n}$. Se tiene

$$c_k = \frac{4k}{n} \quad \Longrightarrow \quad f(c_k) = (2 + c_k)^2 .$$

Como

$$c_k = \frac{4k}{n} = 0 + k\left(\frac{4}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{4}{n} = \frac{b-a}{n}, \\ b &= 4 + a = 4 + 0 = 4. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{4k}{n}\right)^2 \left(\frac{4}{n}\right) = \int_0^4 (2+x)^2 \ dx.$$

Nota que las dos respuestas son equivalentes. En efecto, usando la sustitución $u = 2 + x$ en la integral de la segunda respuesta, se obtiene

$$\int_0^4 (2+x)^2 \ dx = \int_2^6 u^2 \, du.$$

(c) $\displaystyle \lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{\left[4 - \left(-1 + \frac{3k}{n}\right)^2\right]}_{f(c_k)} \underbrace{\left(\frac{3}{n}\right)}_{\Delta x}.$

Sea $c_k = -1 + \dfrac{3k}{n}$. Se tiene

$$c_k = -1 + \frac{3k}{n} \quad \Longrightarrow \quad f(c_k) = 4 - c_k^2.$$

Como

$$c_k = -1 + \frac{3k}{n} = -1 + k\left(\frac{3}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= -1, \\ \Delta x &= \frac{3}{n} = \frac{b-a}{n}, \\ b &= 3 + a = 3 + (-1) = 2. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[4 - \left(-1 + \frac{3k}{n}\right)^2\right] \left(\frac{3}{n}\right) = \int_{-1}^{2} \left(4 - x^2\right) \ dx.$$

(d) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \underbrace{\left[\frac{1}{1+3k/n}\right]}_{f(c_k)} \underbrace{\left(\frac{3}{n}\right)}_{\Delta x}$.

Aquí hay dos respuestas directas:

(d.1) Sea $c_k = \dfrac{3k}{n}$. Se tiene

$$c_k = \frac{3k}{n} \quad \Longrightarrow \quad f(c_k) = \frac{1}{1+c_k}.$$

Como

$$c_k = \frac{3k}{n} = 0 + k\left(\frac{3}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{3}{n} = \frac{b-a}{n}, \\ b &= 3 + a = 3 + 0 = 3. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3/n}{1+3k/n} = \int_0^3 \frac{1}{1+x}\,dx.$$

(d.2) Sea $c_k = 1 + \dfrac{3k}{n}$. Se tiene

$$c_k = 1 + \frac{3k}{n} \quad \Longrightarrow \quad f(c_k) = \frac{1}{c_k}.$$

Como

$$c_k = 1 + \frac{3k}{n} = 1 + k\left(\frac{3}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 1, \\ \Delta x &= \frac{3}{n} = \frac{b-a}{n}, \\ b &= 3 + a = 3 + 1 = 4. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3/n}{1+3k/n} = \int_1^4 \frac{1}{x}\,dx.$$

(e) $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{n} \frac{1}{1+(k/n)^2} = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \underbrace{\left[\frac{1}{1+(k/n)^2}\right]}_{f(c_k)} \underbrace{\left(\frac{1}{n}\right)}_{\Delta x}.$

Sea $c_k = \frac{k}{n}$. Se tiene

$$c_k = \frac{k}{n} \quad \Longrightarrow \quad f(c_k) = \frac{1}{1+c_k^2}.$$

Como

$$c_k = \frac{k}{n} = 0 + k\left(\frac{1}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{1}{n} = \frac{b-a}{n}, \\ b &= 1 + a = 1 + 0 = 1. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+(k/n)^2} = \int_0^1 \frac{1}{1+x^2}\,dx.$$

(f) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{k^4}{n^5} = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \underbrace{\left(\frac{k}{n}\right)^4}_{f(c_k)} \underbrace{\left(\frac{1}{n}\right)}_{\Delta x}.$

Sea $c_k = \frac{k}{n}$. Se tiene

$$c_k = \frac{k}{n} \quad \Longrightarrow \quad f(c_k) = c_k^4.$$

Como

$$c_k = \frac{k}{n} = 0 + k\left(\frac{1}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{1}{n} = \frac{b-a}{n}, \\ b &= 1 + a = 1 + 0 = 1. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^4}{n^5} = \int_0^1 x^4\,dx.$$

(g) $\lim\limits_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^5+\left(\frac{2}{n}\right)^5+\left(\frac{3}{n}\right)^5+...+\left(\frac{n}{n}\right)^5\right] = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\underbrace{\left(\frac{k}{n}\right)^5}_{f(c_k)}\underbrace{\left(\frac{1}{n}\right)}_{\Delta x}.$

Sea $c_k = \frac{k}{n}$. Se tiene

$$c_k = \frac{k}{n} \quad \Longrightarrow \quad f(c_k) = f(c_k) = c_k^5.$$

Como

$$c_k = \frac{k}{n} = 0 + k\left(\frac{1}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{1}{n} = \frac{b-a}{n}, \\ b &= 1 + a = 1 + 0 = 1. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^5+\left(\frac{2}{n}\right)^5+\left(\frac{3}{n}\right)^5+...+\left(\frac{n}{n}\right)^5\right] = \int_0^1 x^5\,dx.$$

(h) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{3/n}{\sqrt{2+5k/n}} = \frac{3}{5}\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\underbrace{\left[\frac{1}{\sqrt{2+5k/n}}\right]}_{f(c_k)}\underbrace{\left(\frac{5}{n}\right)}_{\Delta x}.$

Aquí hay dos respuestas directas:

(h.1) Sea $c_k = \frac{5k}{n}$. Se tiene

$$c_k = \frac{5k}{n} \quad \Longrightarrow \quad f(c_k) = \frac{1}{\sqrt{2+c_k}}.$$

Como

$$c_k = \frac{5k}{n} = 0 + k\left(\frac{5}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{5}{n} = \frac{b-a}{n}, \\ b &= 5 + a = 5 + 0 = 5. \end{aligned}$$

Así,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{3/n}{\sqrt{2+5k/n}} = \frac{3}{5}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{5/n}{\sqrt{2+5k/n}} = \frac{3}{5}\int_0^5 \frac{1}{\sqrt{2+x}}\,dx.$$

(h.2) Sea $c_k = 2 + \frac{5k}{n}$. Se tiene

$$c_k = 2 + \frac{5k}{n} \quad \Longrightarrow \quad f(c_k) = \frac{1}{\sqrt{c_k}}.$$

Como

$$c_k = 2 + \frac{5k}{n} = 2 + k\left(\frac{5}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 2, \\ \Delta x &= \frac{5}{n} = \frac{b-a}{n}, \\ b &= 5 + a = 5 + 2 = 7. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3/n}{\sqrt{2+5k/n}} = \frac{3}{5} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{5/n}{\sqrt{2+5k/n}} = \frac{3}{5} \int_2^7 \frac{1}{\sqrt{x}}\, dx.$$

(i) $$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3/n}{\left(\sqrt{2+3k/n}\right)(3k/n)} = \lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{\left[\frac{1}{\left(\sqrt{2+3k/n}\right)(3k/n)}\right]}_{f(c_k)} \underbrace{\left(\frac{3}{n}\right)}_{\Delta x}.$$

Sea $c_k = \frac{3k}{n}$. Se tiene

$$c_k = \frac{3k}{n} \quad \Longrightarrow \quad f(c_k) = \frac{1}{\left(\sqrt{2+c_k}\right) c_k}.$$

Como

$$c_k = \frac{3k}{n} = 0 + k\left(\frac{3}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{3}{n} = \frac{b-a}{n}, \\ b &= 3 + a = 3 + 0 = 3. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3/n}{\left(\sqrt{2+3k/n}\right)(3k/n)} = \int_0^3 \frac{1}{\left(\sqrt{2+x}\right) x}\, dx.$$

11. Para calcular la integral $\int_0^1 (2x)\ dx$ utilizando sumas de Riemann, observamos que

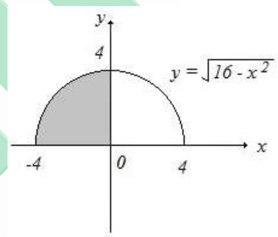
$$f(c_k) = 2c_k,$$
$$a = 0, \quad b = 1 \quad \therefore \quad \Delta x = \frac{b-a}{n} = \frac{1}{n}$$
$$\therefore \quad c_k = a + k\Delta x = 0 + k\left(\frac{1}{n}\right) = \frac{k}{n}.$$

De esta manera,

$$\begin{aligned} \int_0^1 (2x)\ dx &= \lim_{\|P\|\to 0} \sum_{k=1}^{n} (2c_k)\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{2k}{n}\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{2}{n^2}\sum_{k=1}^{n} k\right) \\ &= \lim_{n\to\infty}\left(\frac{2}{n^2}\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1. \end{aligned}$$

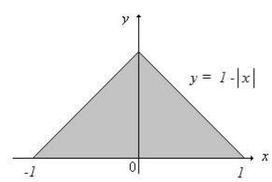
12. (a) La integral $\int_{-4}^{0} \sqrt{16-x^2}\ dx$ representa la cuarta parte del área de un círculo con centro en el origen y radio igual a 4:

$$\int_{-4}^{0} \sqrt{16-x^2}\ dx = \frac{1}{4}\left(\pi \cdot 4^2\right) = 4\pi.$$



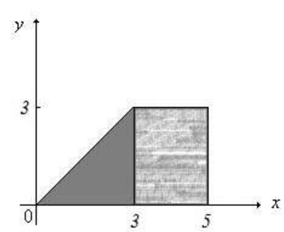
(b) La integral $\int_{-1}^{1} (1-|x|)\ dx$ representa el área de un triángulo isósceles con base 2 y altura 1:

$$\int_{-1}^{1} (1-|x|)\ dx = \frac{2(1)}{2} = 1.$$



(c) Si $f(x) = \begin{cases} x, & 0 \le x < 3 \\ 3, & 3 \le x \le 5. \end{cases}$, la integral $\int_0^5 f(x)\ \ dx$ representa el área del trapecio mostrado en la figura:

$$\int_0^5\ f(x)\ dx = \frac{3(3)}{2} + 2(3) = \frac{21}{2}.$$



13. Sea f continua y tal que $\int\limits_1^5 f(x)\ dx = -1$ y $\int\limits_3^5 f(x)\ dx = 4.$

(a) $\int\limits_1^5 f(u)\ du = \int\limits_1^5 f(x)\ dx = -1.$

(b) $\int\limits_1^1 f(x)\ dx = \int\limits_1^1 f(x)\ dx = 0.$

(c) $\int\limits_5^1 [-3f(t)]\ dt = -3\int\limits_5^1 f(t)\ dt = 3\int\limits_1^5 f(t)\ dt = 3(-1) = -3.$

(d) $\int\limits_1^3 f(x)\ dx = \int\limits_1^5 f(x)\ dx - \int\limits_3^5 f(x)\ dx = (-1) - (4) = -5.$

14. (a) Si f es continua y $f(x) \geq 0$ para todo $x \in [a,b]$, entonces $\int_a^b f(x)\ dx \geq 0$.
Verdadera.

(b) Si $\int_a^b f(x)\ dx \geq 0$, entonces $f(x) \geq 0$ para todo $x \in [a,b]$.
Falsa. Ejemplo: $\int_{-1}^2 x\ dx = \dfrac{3}{2} > 2$, pero $f(x) = x < 0$ para algunos $x \in [-1,2]$.

(c) Si $\int_a^b f(x)\ dx = 0$, entonces $f(x) = 0$ para todo $x \in [a,b]$.
Falsa. Ejemplo: $\int_{-1}^1 x\ dx = 0$, pero $f(x) \neq 0$ para algunos $x \in [-1,1]$.

(d) Si $f(x) \geq 0$ y $\int_a^b f(x)\ dx = 0$, entonces $f(x) = 0$ para todo $x \in [a,b]$.
Falsa. Ejemplo, considera la función $f(x) \geq 0$ definida por

$$f(x) = \begin{cases} 0, & \text{si } x \in [1,2) \cup (2,3], \\ 5, & \text{si } x = 2. \end{cases}$$

Claramente $\int_1^3 f(x)\ dx = 0$, pero $f(x) \neq 0$.

(e) Si $\int_a^b f(x)\ dx > \int_a^b g(x)\ dx$, entonces $\int_a^b [f(x) - g(x)]\ dx > 0$.
Verdadera.

(f) Si f y g son continuas y $f(x) > g(x)$ para todo $x \in [a,b]$, entonces $\left|\int_a^b f(x)\ dx\right| > \left|\int_a^b g(x)\ dx\right|$.
Falsa. Ejemplo: Sean $f(x) = x$ y $g(x) = -2x$. Claramente, $f(x) > g(x)$ en $[1,2]$. Sin embargo, $\dfrac{3}{2} = \left|\int_1^2 x\ dx\right| < \left|\int_1^2 (-2x)\ dx\right| = 3$.

15. El integrando de $\int\limits_1^3 \dfrac{1}{1+x^2}\ dx$ es la función continua $f(x) = \dfrac{1}{1+x^2}$. Los valores mínimo y máximo de f en el intervalo $[1,3]$ son, respectivamente,

$$\min f = \frac{1}{1+3^2} = \frac{1}{10}, \qquad \max f = \frac{1}{1+1} = \frac{1}{2}.$$

Por la desigualdad max-min se tiene

$$\left(\frac{1}{10}\right)(3-1) \leq \int_1^3 \frac{1}{1+x^2}\,dx \leq \left(\frac{1}{2}\right)(3-1),$$

es decir,

$$\frac{1}{5} \leq \int_1^3 \frac{1}{1+x^2}\,dx \leq 1.$$

Por lo tanto, el valor de la integral está entre $1/5$ y 1.

16. (a) $\int_1^3 \pi\,dx = \pi \int_1^3 dx = \pi(3-1) = 2\pi.$

(b) $\int_{-1/2}^{1/2} (-3x)\,dx = -3\int_{-1/2}^{1/2} x\,dx = -3\left[\frac{(1/2)^2}{2} - \frac{(-1/2)^2}{2}\right] = 0.$

(c)

$$\begin{aligned}\int_{-1}^{0}\left[x^2 - \frac{x}{2} + \frac{1}{2}\right] dx &= \int_{-1}^{0} x^2\,dx - \frac{1}{2}\int_{-1}^{0} x\,dx + \frac{1}{2}\int_{-1}^{0} dx \\ &= \left[\frac{0^3}{3} - \frac{(-1)^3}{3}\right] - \frac{1}{2}\left[\frac{0^2}{2} - \frac{(-1)^2}{2}\right] + \frac{1}{2}\left[0-(-1)\right] = \frac{13}{12}.\end{aligned}$$

CÁLCULO III
TAREA 3 - SOLUCIONES
TEOREMA FUNDAMENTAL DEL CÁLCULO. SUSTITUCIÓN EN INTEGRAL DEFINIDA
(Temas 1.3-1.4)

1. (a) $G(x) = \int_1^x \sqrt{1+t^4}\, dt.$

$$\frac{dG(x)}{dx} = \frac{d}{dx}\int_1^x \sqrt{1+t^4}\, dt = \sqrt{1+x^4}.$$

(b) $G(x) = \int_x^1 \sqrt{1+t^4}\, dt.$

$$\frac{dG(x)}{dx} = \frac{d}{dx}\int_x^1 \sqrt{1+t^4}\, dt = \frac{d}{dx}\left(-\int_1^x \sqrt{1+t^4}\, dt\right) = -\sqrt{1+x^4}.$$

(c) $G(x) = \int_{-1}^1 \sqrt{1+t^4}\, dt.$

$$\frac{dG(x)}{dx} = \frac{d}{dx}\underbrace{\int_{-1}^1 \sqrt{1+t^4}\, dt}_{\text{cons tan te}} = 0.$$

(d) $G(x) = \int_{-1}^1 x^2\sqrt{1+t^4}\, dt.$

$$\begin{aligned}\frac{dG(x)}{dx} &= \frac{d}{dx}\int_{-1}^1 x^2\sqrt{1+t^4}\, dt = \frac{d}{dx}(x^2 \underbrace{\int_{-1}^1 \sqrt{1+t^4}\, dt)}_{\text{cons tan te}} \\ &= \left(\frac{dx^2}{dx}\right)\int_{-1}^1 \sqrt{1+t^4}\, dt = 2x\int_{-1}^1 \sqrt{1+t^4}\, dt.\end{aligned}$$

(e) $G(x) = \int_{-1}^x x^2\sqrt{1+t^4}\, dt.$

$$\begin{aligned}\frac{dG(x)}{dx} &= \frac{d}{dx}\int_{-1}^x x^2\sqrt{1+t^4}\, dt = \frac{d}{dx}\left(x^2\int_{-1}^x \sqrt{1+t^4}\, dt\right) \\ &= x^2\left(\frac{d}{dx}\int_{-1}^x \sqrt{1+t^4}\, dt\right) + \left(\frac{dx^2}{dx}\right)\int_{-1}^x \sqrt{1+t^4}\, dt \\ &= x^2\sqrt{1+x^4} + 2x\int_{-1}^x \sqrt{1+t^4}\, dt.\end{aligned}$$

(f) $G(x)=\int_0^{\operatorname{sen} x}\left(u^2+\operatorname{sen} u\right)\,du.$

$$\begin{aligned}\frac{dG(x)}{dx} &= \frac{d}{dx}\int_0^{\operatorname{sen} x}\left(u^2+\operatorname{sen} u\right)\,du=\left[(\operatorname{sen} x)^2+\operatorname{sen}(\operatorname{sen} x)\right]\frac{d\operatorname{sen} x}{dx}\\ &= \left[\operatorname{sen}^2 x+\operatorname{sen}(\operatorname{sen} x)\right]\cos x.\end{aligned}$$

(g) $G(x)=\int_x^{x^3}\sqrt{1+t^4}\,dt.$

Usando la regla de Leibnitz se tiene

$$\begin{aligned}\frac{dG(x)}{dx} &= \frac{d}{dx}\int_x^{x^3}\sqrt{1+t^4}\,dt=\sqrt{1+(x^3)^4}\,\frac{d\,x^3}{dx}-\sqrt{1+x^4}\,\frac{d\,x}{dx}\\ &= \sqrt{1+x^{12}}\left(3x^2\right)-\sqrt{1+x^4}.\end{aligned}$$

2. (a) $r(\theta)=\int_{e^{\sqrt{\theta}}}^{e^{3\theta}}\ln t\,dt.$

$$\begin{aligned}\frac{dr(\theta)}{d\theta} &= \frac{d}{d\theta}\int_{e^{\sqrt{\theta}}}^{e^{3\theta}}\ln t\,dt=\ln\left(e^{3\theta}\right)\frac{de^{3\theta}}{d\theta}-\ln\left(e^{\sqrt{\theta}}\right)\frac{de^{\sqrt{\theta}}}{d\theta}=3\theta\left(3e^{3\theta}\right)-\sqrt{\theta}\left(\frac{1}{2\sqrt{\theta}}e^{\sqrt{\theta}}\right)\\ &= 9\theta e^{3\theta}-\frac{1}{2}e^{\sqrt{\theta}}.\end{aligned}$$

(b) $\alpha(u)=\int_0^{\ln u} sen(e^x)\,dx.$

$$\frac{d\alpha(u)}{du}=\frac{d}{du}\int_0^{\ln u}\operatorname{sen}\left(e^x\right)\,dx=\operatorname{sen}\left(e^{\ln u}\right)\frac{d\ln u}{du}=(\operatorname{sen} u)\left(\frac{1}{u}\right)=\frac{\operatorname{sen} u}{u}.$$

(c) $G(x)=\int_0^{2\ln x}\left(e^t+t^2\right)\,dt,\quad x>1.$

$$\begin{aligned}\frac{dG(x)}{dx} &= \frac{d}{dx}\int_0^{2\ln x}\left(e^t+t^2\right)\,dt=\left[e^{2\ln x}+(2\ln x)^2\right]\frac{d\,2\ln x}{dx}\\ &= \left[e^{\ln x^2}+4(\ln x)^2\right]\frac{2}{x}=\frac{2}{x}\left[x^2+4\ln^2 x\right].\end{aligned}$$

(d) $x(y)=\int_0^{y} y^2\left(\frac{1}{1+s^3}\right)\,ds.$

$$\begin{aligned}\frac{dx(y)}{dy} &= \frac{d}{dy}\int_0^{y} y^2\left(\frac{1}{1+s^3}\right)\,ds=\frac{d}{dy}\left(y^2\int_0^{y}\frac{1}{1+s^3}\,ds\right)\\ &= \left(\frac{dy^2}{dy}\right)\int_0^{y}\frac{1}{1+s^3}\,ds+y^2\left(\frac{d}{dy}\int_0^{y}\frac{1}{1+s^3}\,ds\right)\\ &= 2y\int_0^{y}\frac{1}{1+s^3}\,ds+\frac{y^2}{1+y^3}.\end{aligned}$$

(e) $x(t) = \int\limits_0^t e^{\left(t^2 - z^2\right)}\, dz.$

$$\begin{aligned}\frac{dx(t)}{dt} &= \frac{d}{dt}\int\limits_0^t e^{\left(t^2-z^2\right)}\, dz = \frac{d}{dt}\left(e^{t^2}\int\limits_0^t e^{-z^2}\, dz\right)\\ &= \left(\frac{de^{t^2}}{dt}\right)\int\limits_0^t e^{-z^2}\, dz + e^{t^2}\left(\frac{d}{dt}\int\limits_0^t e^{-z^2}\, dz\right)\\ &= \left(2te^{t^2}\right)\int\limits_0^t e^{-z^2}\, dz + e^{t^2}e^{-t^2} = 2t\int\limits_0^t e^{\left(t^2-z^2\right)}\, dz + 1.\end{aligned}$$

Nota que la respuesta también se puede escribir como

$$\frac{dx(t)}{dt} = 2t\, x(t) + 1.$$

3. Sea $f(x) = \int\limits_0^{1/x} \frac{1}{t^2+1}\, dt + \int\limits_0^x \frac{1}{t^2+1}\, dt$, con $x > 0$. Como

$$\begin{aligned}\frac{df(x)}{dx} &= \frac{d}{dx}\int\limits_0^{1/x}\frac{1}{t^2+1}\, dt + \frac{d}{dx}\int\limits_0^x \frac{1}{t^2+1}\, dt = \frac{1}{(1/x)^2+1}\cdot\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x^2+1}\\ &= \frac{x^2}{1+x^2}\left(-\frac{1}{x^2}\right) + \frac{1}{x^2+1} = 0,\end{aligned}$$

por lo tanto f es constante para todo $x > 0$.

4. Sea $G(x) = \int\limits_a^x f(t)\, dt$, con a una constante.

(a) $\frac{dG(x)}{dx} = \frac{d}{dx}\int\limits_a^x f(t)\, dt = f(x)$.

(b) $G(x^2) = \int\limits_a^{x^2} f(t)\, dt$.

(c) $\frac{dG(x^2)}{dx} = \frac{d}{dx}\int\limits_a^{x^2} f(t)\, dt = f(x^2)\left(\frac{dx^2}{dx}\right) = 2x\, f(x^2)$.

5. Sea $F(x) = \int\limits_4^{x^2} e^{\sqrt{t}}\, dt$, para todo $x \geq 2$.

(a) $F(2) = \int\limits_4^4 e^{\sqrt{t}}\, dt = 0.$

(b) $F'(x) = \frac{d}{dx}\int\limits_4^{x^2} e^{\sqrt{t}}\, dt = e^{\sqrt{x^2}}\left(\frac{dx^2}{dx}\right) = 2xe^{|x|} = 2xe^x$. Nota que $\sqrt{x^2} = |x| = x$, para $x \geq 2$.
$F'(2) = 4e^2$

(c) $F''(x) = \dfrac{d}{dx}\left(2xe^{x}\right) = 2xe^{x} + 2e^{x} = 2\left(x+1\right)e^{x}$.
$F''(2) = 6e^{2}$

6. Sea f tal que $f' > 0$ y $f\left(1\right) = 0$. Sea $g\left(x\right) = \int\limits_{0}^{x} f\left(t\right)\ dt$.

(a) g es una funcion diferenciable de x.
Verdadero.
Como f es diferenciable, por lo tanto f es continua. Por el Teorema Fundamental de Cálculo, $g\left(x\right)$ es diferenciable (con $\dfrac{dg}{dx} = f$).

(b) g es una funcion continua de x.
Verdadero.
Como g es diferenciable, por lo tanto g es continua.

(c) La gráfica de g tiene una tangente horizontal en $x = 1$.
Verdadero.
$\dfrac{dg(x)}{dx} = f\left(x\right) \quad \therefore \quad \left.\dfrac{dg(x)}{dx}\right|_{x=1} = f\left(1\right) = 0 \quad \therefore$ en $x = 1$ la tangente de g es horizontal.

(d) g tiene un máximo local en $x = 1$.
Falso.
$g''(x) = f'(x) > 0 \;\therefore\; g$ es convexa $\forall x \;\therefore\; g$ tiene un mínimo local en $x = 1$.

(e) La gráfica de $\dfrac{dg}{dx}$ cruza el eje x en $x = 1$.
Verdadero.
$g'(1) = \left.\dfrac{dg(x)}{dx}\right|_{x=1} = f\left(1\right) = 0.$

7. Reescribimos el precio $P\left(t\right)$, de la siguiente manera:

$$P\left(t\right) = \int\limits_{t}^{20} v\left(s\right)e^{r(t-s)}\ ds = e^{rt}\int\limits_{t}^{20} v\left(s\right)e^{-rs}\ ds = -e^{rt}\int\limits_{20}^{t} v\left(s\right)e^{-rs}\ ds.$$

En ese caso,

$$\begin{aligned}
\frac{dP\left(t\right)}{dt} &= -\left(\frac{de^{rt}}{dt}\right)\int\limits_{20}^{t} v\left(s\right)e^{-rs}\ ds - e^{rt}\frac{d}{dt}\int\limits_{20}^{t} v\left(s\right)e^{-rs}\ ds \\
&= -re^{rt}\int\limits_{20}^{t} v\left(s\right)e^{-rs}\ ds - e^{rt}\left(v\left(t\right)e^{-rt}\right) \\
&= -r\int\limits_{20}^{t} v\left(s\right)e^{r(t-s)}\ ds - v\left(t\right) \\
&= r\int\limits_{t}^{20} v\left(s\right)e^{r(t-s)}\ ds - v\left(t\right) \\
&= rP\left(t\right) - v\left(t\right).
\end{aligned}$$

8. De acuerdo con el Teorema Fundamental del Cálculo,

$$\frac{dF(x)}{dx} = \sqrt{5+e^{x^2}} \quad \Longrightarrow \quad F(x) = \int_a^x \sqrt{5+e^{t^2}}\,dt,$$

con a una constante. En particular,

$$F(2) = \int_a^2 \sqrt{5+e^{t^2}}\,dt,$$

de modo que

$$\begin{aligned} F(x) - F(2) &= \int_a^x \sqrt{5+e^{t^2}}\,dt - \int_a^2 \sqrt{5+e^{t^2}}\,dt \\ &= \int_a^x \sqrt{5+e^{t^2}}\,dt + \int_2^a \sqrt{5+e^{t^2}}\,dt \\ &= \int_2^a \sqrt{5+e^{t^2}}\,dt + \int_a^x \sqrt{5+e^{t^2}}\,dt = \int_2^x \sqrt{5+e^{t^2}}\,dt. \end{aligned}$$

De esta manera,

$$F(x) = \int_2^x \sqrt{5+e^{t^2}}\,dt + F(2).$$

Como $F(2) = 7$, por lo tanto

$$F(x) = \int_2^x \sqrt{5+e^{t^2}}\,dt + 7.$$

En efecto,

$$\begin{aligned} F(2) &= \int_2^2 \sqrt{5+e^{t^2}}\,dt + 7 = 0 + 7 = 7, \\ \frac{dF(x)}{dx} &= \frac{d}{dx}\left(\int_2^x \sqrt{5+e^{t^2}}\,dt + 7\right) = \sqrt{5+e^{x^2}}. \end{aligned}$$

9. De acuerdo con el Teorema Fundamental del Cálculo,

$$\frac{dx(t)}{dt} = t^2 f(t) \quad \Longrightarrow \quad x(t) = \int_a^t u^2 f(u)\,du,$$

con a una constante. En particular,

$$x(3) = \int_a^3 u^2 f(u)\,du,$$

de modo que

$$x(t) - x(3) = \int_a^t u^2 f(u)\,du - \int_a^3 u^2 f(u)\,du = \int_3^t u^2 f(u)\,du.$$

De esta manera,

$$x(t) = \int_3^t u^2 f(u)\ du + x(3).$$

Por último, como $x(3) = 5$, por lo tanto

$$x(t) = \int_3^t u^2 f(u)\ du + 5.$$

10. De acuerdo con el Teorema Fundamental del Cálculo,

$$f(t) = \frac{dF(t)}{dt} \quad \Longrightarrow \quad F(t) = \int_a^t f(u)\ du,$$

con a una constante. En particular,

$$F(t_0) = \int_a^{t_0} f(u)\ du,$$

de modo que

$$F(t) - F(t_0) = \int_a^t f(u)\ du - \int_a^{t_0} f(u)\ du = \int_{t_0}^t f(u)\ du.$$

De esta manera,

$$F(t) = \int_{t_0}^t f(u)\ du + F(t_0).$$

Por último, como $F(t_0) = K$, por lo tanto,

$$F(t) = \int_{t_0}^t f(u)\ du + K.$$

11. (a)

$$\begin{aligned} \int_{-8}^{-1} \left(5x^{2/3} - 5 - \frac{8}{x^2}\right) dx &= 5\left[\frac{x^{5/3}}{\frac{5}{3}}\right]_{-8}^{-1} - 5\left[x\right]_{-8}^{-1} + 8\left[\frac{1}{x}\right]_{-8}^{-1} \\ &= 3\left[(-1)^{5/3} - (-8)^{5/3}\right] - 5\left[(-1) - (-8)\right] + 8\left[\frac{1}{(-1)} - \frac{1}{(-8)}\right] \\ &= 3\left[(-1) - (-32)\right] - 5\left[7\right] + 8\left[-1 + \frac{1}{8}\right] = 51. \end{aligned}$$

(b)

$$\begin{aligned} \int_{-2}^{-1} \frac{y^5 + 1}{y^3}\ dy &= \int_{-2}^{-1} \left(y^2 + y^{-3}\right)\ dy = \left[\frac{y^3}{3} - \frac{1}{2y^2}\right]_{-2}^{-1} \\ &= \left(-\frac{1}{3} - \frac{1}{2}\right) - \left(-\frac{8}{3} - \frac{1}{8}\right) = \frac{7}{3} - \frac{3}{8} = \frac{47}{24}. \end{aligned}$$

(c)

$$\int_0^1 \left(x^2+2x\right)^2\ dx = \int_0^1 \left(x^4+4x^3+4x^2\right)\ dx = \left[\frac{x^5}{5}+x^4+\frac{4x^3}{3}\right]_0^1 = \frac{38}{15}.$$

(d)

$$\begin{aligned}\int_{-\ln 3}^{\ln 3} \left(e^x+1\right)\ dx &= \left[e^x+x\right]_{-\ln 3}^{\ln 3} = \left(e^{\ln 3}+\ln 3\right)-\left(e^{-\ln 3}+(-\ln 3)\right)\\ &= \left(e^{\ln 3}+\ln 3\right)-\left(\frac{1}{e^{\ln 3}}-\ln 3\right) = (3+\ln 3)-\left(\frac{1}{3}-\ln 3\right)\\ &= \frac{8}{3}+2\ln 3 = \frac{8}{3}+\ln 9.\end{aligned}$$

(e) Como

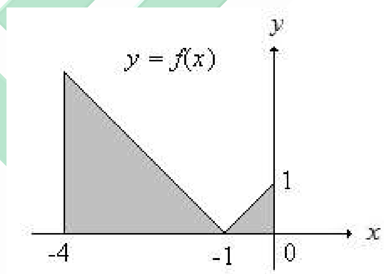
$$|x+1| = \begin{cases} x+1, & x\geq -1,\\ -x-1, & x<-1,\end{cases}$$

por lo tanto,

$$\therefore \quad \int_{-4}^{0} |x+1|\ dx = \int_{-4}^{-1} (-x-1)\ dx + \int_{-1}^{0} (x+1)\ dx = \left[-\frac{x^2}{2}-x\right]_{-4}^{-1}+\left[\frac{x^2}{2}+x\right]_{-1}^{0} = 5.$$

O bien, por sustitución,

$$\int_{-4}^{0} |x+1|\ dx = \int_{-3}^{1} |u|\ du = -\int_{-3}^{0} u\ du + \int_{0}^{1} u\ du = -\left[\frac{u^2}{2}\right]_{-3}^{0}+\left[\frac{u^2}{2}\right]_{0}^{1} = 5.$$



12. (a) $\int_1^3 \sqrt{6-2x}\ dx.$

Sea $u=6-2x$. Se tiene

$$u=6-2x \quad\Longrightarrow\quad du=-2\ dx,\ u(1)=4,\ u(3)=0.$$

Por lo tanto,

$$\begin{aligned}\int_1^3 \sqrt{6-2x}\ dx &= -\frac{1}{2}\int_1^3 \sqrt{6-2x}\ (-2\ dx) = -\frac{1}{2}\int_4^0 \sqrt{u}\ du = \frac{1}{2}\int_0^4 \sqrt{u}\ du\\ &= \frac{1}{3}\left[u^{3/2}\right]_0^4 = \frac{1}{3}\left[4^{3/2}-0\right] = \frac{8}{3}.\end{aligned}$$

(b) $\displaystyle\int_1^x \frac{t^2}{\sqrt{t^3+3}}\, dt.$

Sea $u = t^3 + 3$. Se tiene

$$u = t^3 + 3 \quad \Longrightarrow \quad du = 3t^2\, dt,\ u(1) = 4,\ u(x) = x^3 + 3.$$

Por lo tanto,

$$\begin{aligned} \int_1^x \frac{t^2}{\sqrt{t^3+3}}\, dt &= \frac{1}{3}\int_1^x \frac{1}{\sqrt{t^3+3}}\, 3t^2\, dt = \frac{1}{3}\int_{\substack{4\\4}}^{x^3+3} \frac{du}{\sqrt{u}} \\ &= \frac{2}{3}\left[\sqrt{u}\right]_4^{x^3+3} = \frac{2}{3}\left[\sqrt{x^3+3} - 2\right]. \end{aligned}$$

(c) $\displaystyle\int_1^3 \frac{x^2+1}{x^3+3x}\, dx.$

Sea $u = x^3 + 3x$. Se tiene

$$u = x^3 + 3x \quad \Longrightarrow \quad du = \left(3x^2 + 3\right)\, dx,\ u(1) = 4,\ u(3) = 36.$$

Por lo tanto,

$$\begin{aligned} \int_1^3 \frac{x^2+1}{x^3+3x}\, dx &= \frac{1}{3}\int_1^3 \frac{1}{x^3+3x}\left(3x^2+3\right)\, dx = \frac{1}{3}\int_4^{36} \frac{du}{u} \\ &= \frac{1}{3}\left[\ln|u|\right]_4^{36} = \frac{1}{3}\left(\ln 36 - \ln 4\right) = \frac{1}{3}\ln\left(\frac{36}{4}\right) = \frac{1}{3}\ln 9. \end{aligned}$$

(d) $\displaystyle\int_1^4 \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)^2}.$

Sea $u = 1 + \sqrt{x}$. Se tiene

$$u = 1 + \sqrt{x} \quad \Longrightarrow \quad du = \frac{1}{2\sqrt{x}}\, dx,\ u(1) = 2,\ u(4) = 3.$$

Por lo tanto,

$$\begin{aligned} \int_1^4 \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)^2} &= \int_1^4 \frac{1}{\left(1+\sqrt{x}\right)^2}\left(\frac{1}{2\sqrt{x}}\right) dx = \int_2^3 \frac{du}{u^2} = \left[-\frac{1}{u}\right]_2^3 \\ &= -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}. \end{aligned}$$

(e) $\displaystyle\int_0^3 \frac{x}{\sqrt{1+x}}\, dx.$

Sea $u = \sqrt{1+x}$. Se tiene

$$u = \sqrt{1+x} \quad \Longrightarrow \quad x = u^2 - 1 \quad \Longrightarrow \quad dx = 2u\, du,\ u(0) = 1,\ u(3) = 2.$$

Por lo tanto,

$$\begin{aligned}\int_0^3 \frac{x}{\sqrt{1+x}}\,dx &= \int_1^2 \frac{u^2-1}{u}\,2u\,du = 2\int_1^2 \left(u^2-1\right)\,du = 2\left[\frac{u^3}{3}-u\right]_1^2 \\ . &= 2\left[\left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-1\right)\right] = \frac{8}{3}.\end{aligned}$$

(f) $\int\limits_{-4\ln 3}^{0} \sqrt{e^x}\,dx = \int\limits_{-4\ln 3}^{0} e^{x/2}\,dx.$

Sea $u = \dfrac{x}{2}$. Se tiene

$$u = \frac{x}{2} \quad\Longrightarrow\quad du = \frac{1}{2}\,dx,\ u(-4\ln 3) = -2\ln 3,\ u(0) = 0.$$

Por lo tanto,

$$\begin{aligned}\int\limits_{-4\ln 3}^{0} \sqrt{e^x}\,dx &= 2\int\limits_{-4\ln 3}^{0} e^{x/2}\left(\frac{1}{2}\,dx\right) = 2\int\limits_{-2\ln 3}^{0} e^u\,du = 2\,\left[e^u\right]_{-2\ln 3}^{0} \\ &= 2(e^0 - e^{-2\ln 3}) = 2(1-e^{\ln 3^{-2}}) = 2(1-\frac{1}{9}) = \frac{16}{9}.\end{aligned}$$

(g) $\int\limits_0^{\ln 2} \dfrac{e^{3x}\,dx}{2e^{3x}-1}.$

Sea $u = 2e^{3x}-1$. Se tiene

$$u = 2e^{3x}-1 \quad\Longrightarrow\quad du = 6e^{3x}\,dx,\ u(0)=1,\ u(\ln 2) = 2e^{3\ln 2}-1 = 2e^{\ln 8}-1 = 15.$$

Por lo tanto,

$$\begin{aligned}\int\limits_0^{\ln 2} \frac{e^{3x}\,dx}{2e^{3x}-1} &= \frac{1}{6}\int\limits_0^{\ln 2} \frac{6e^{3x}\,dx}{2e^{3x}-1} = \frac{1}{6}\int\limits_1^{15} \frac{du}{u} = \frac{1}{6}\left[\ln|u|\right]_1^{15} = \frac{1}{6}(\ln 15 - \ln 1) \\ &= \frac{1}{6}\ln 15.\end{aligned}$$

(h) $\int\limits_1^e \dfrac{dx}{x\,(\ln x+1)}.$

Sea $u = \ln x + 1$. Se tiene

$$u = \ln x + 1 \quad\Longrightarrow\quad du = \frac{1}{x}\,dx,\ u(1) = \ln 1 + 1 = 1,\ u(e) = \ln e + 1 = 2.$$

Por lo tanto,

$$\int\limits_1^e \frac{dx}{x\,(\ln x+1)} = \int\limits_1^e \frac{1}{\ln x+1}\left(\frac{1}{x}\right)dx = \int\limits_1^2 \frac{du}{u} = \left[\ln|u|\right]_1^2 = \ln 2 - \ln 1 = \ln 2.$$

(i) $\displaystyle\int_{\pi/8}^{\pi/4} \frac{\cos(2\theta)}{\operatorname{sen}^3(2\theta)}\, d\theta.$

Sea $u = sen(2\theta)$. Se tiene

$$u = \operatorname{sen}(2\theta) \quad \Longrightarrow \quad du = 2\cos(2\theta)\ d\theta,\ u(\frac{\pi}{8}) = \frac{1}{\sqrt{2}},\ u(\frac{\pi}{4}) = 1.$$

Por lo tanto,

$$\begin{aligned}\int_{\pi/8}^{\pi/4} \frac{\cos(2\theta)}{\operatorname{sen}^3(2\theta)}\, d\theta &= \frac{1}{2}\int_{\pi/8}^{\pi/4} \frac{2\cos(2\theta)\ d\theta}{[\operatorname{sen}(2\theta)]^3} = \frac{1}{2}\int_{1/\sqrt{2}}^{1} \frac{du}{u^3} \\ &= \frac{1}{2}\left[-\frac{1}{2u^2}\right]_{1/\sqrt{2}}^{1} = \frac{1}{2}\left(-\frac{1}{2}+1\right) = \frac{1}{4}.\end{aligned}$$

(j) $\displaystyle\int_{0}^{\pi/2} \frac{5\operatorname{sen} x\ \cos x}{(1+\operatorname{sen}^2 x)^2}\, dx.$

Sea $u = 1{+}sen^2 x$. Se tiene

$$u = 1+\operatorname{sen}^2 x \quad \Longrightarrow \quad du = 2\operatorname{sen} x\ \cos x\ dx,\ u(0) = 1,\ u(\frac{\pi}{2}) = 2.$$

Por lo tanto,

$$\begin{aligned}\int_{0}^{\pi/2} \frac{5\operatorname{sen} x\ \cos x}{(1+\operatorname{sen}^2 x)^2}\, dx &= \frac{5}{2}\int_{0}^{\pi/2} \frac{2\operatorname{sen} x\ \cos x\ dx}{(1+\operatorname{sen}^2 x)^2} = \frac{5}{2}\int_{1}^{2} \frac{du}{u^2} \\ &= \frac{5}{2}\left[-\frac{1}{u}\right]_{1}^{2} = \frac{5}{2}(\left(-\frac{1}{2}+1\right) = \frac{5}{4}.\end{aligned}$$

13. (a) Sea $u = x + \lambda$. Se tiene

$$u = x+\lambda \quad \Longrightarrow \quad du = dx,\ u(a) = a+\lambda,\ u(b) = b+\lambda,\ x = u-\lambda.$$

Por lo tanto,

$$\int_a^b f(x)\ dx = \int_{a+\lambda}^{b+\lambda} f(u-\lambda)du = \int_{a+\lambda}^{b+\lambda} f(x-\lambda)dx.$$

La última igualdad del lado derecho es válida, ya que en una integral definida la variable de integración es una variable muda (es indistinto llamarla u o x).

(b) Sea $u = \lambda x$. Se tiene

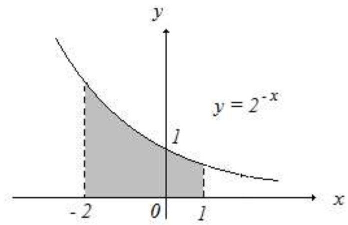
$$u = \lambda x \quad \Longrightarrow \quad du = \lambda\ dx,\ u(a) = \lambda a,\ u(b) = \lambda b,\ x = \frac{u}{\lambda}.$$

Por lo tanto,

$$\int_a^b f(x)\ dx = \frac{1}{\lambda}\int_a^b f(x)\ \lambda dx = \frac{1}{\lambda}\int_{\lambda a}^{\lambda b} f\left(\frac{u}{\lambda}\right)\ du = \frac{1}{\lambda}\int_{\lambda a}^{\lambda b} f\left(\frac{x}{\lambda}\right)\ dx.$$

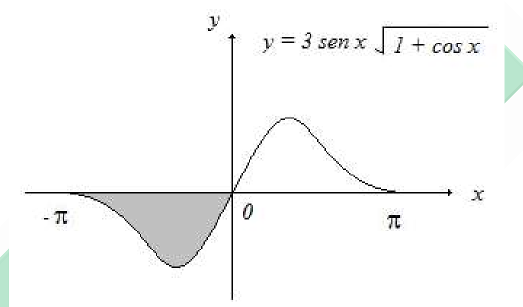
CÁLCULO III
TAREA 4 - SOLUCIONES
ÁREA. VALOR PROMEDIO. LONGITUD DE CURVA
(Tema 1.5)

1. (a) $y = 2^{-x}, \quad -2 \leq x \leq 1.$



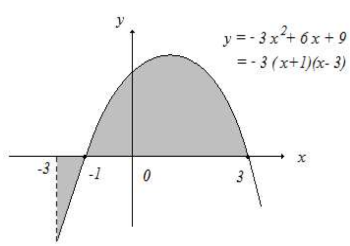
$$A = \int_{-2}^{1} \left|2^{-x}\right| \, dx = \int_{-2}^{1} 2^{-x} \, dx = \int_{-1}^{2} 2^{u} \, du = \frac{1}{\ln 2} \left[2^u\right]_{-1}^{2} = \frac{1}{\ln 2} \left[2^2 - 2^{-1}\right]_{-1}^{2} = \frac{7}{2\ln 2}.$$

(b) $y = 3 \operatorname{sen} x \sqrt{1 + \cos x}, \quad -\pi \leq x \leq 0.$



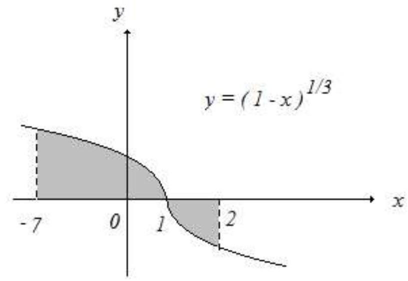
$$\begin{aligned} A &= \int_{-\pi}^{0} \left|3 \operatorname{sen} x \sqrt{1 + \cos x}\right| \, dx = 3 \int_{-\pi}^{0} \left|\operatorname{sen} x\right| \sqrt{1 + \cos x} \, dx \\ &= 3 \int_{-\pi}^{0} (-\operatorname{sen} x) \sqrt{1 + \cos x} \, dx = 3 \int_{0}^{2} \sqrt{u} \, du = 2 \left[u^{3/2}\right]_{0}^{2} = 2^{5/2}. \end{aligned}$$

(c) $y = -3x^2 + 6x + 9, \quad -3 \leq x \leq 3.$



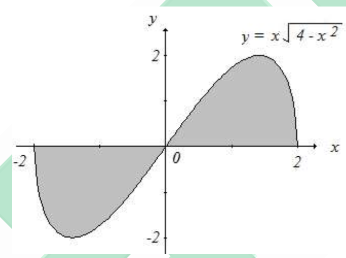
$$\begin{aligned} A &= \int_{-3}^{3} \left|-3x^2 + 6x + 9\right| \, dx = \int_{-3}^{-1} \left(3x^2 - 6x - 9\right) \, dx + \int_{-1}^{3} \left(-3x^2 + 6x + 9\right) \, dx \\ &= \left[x^3 - 3x^2 - 9x\right]_{-3}^{-1} + \left[-x^3 + 3x^2 + 9x\right]_{-1}^{3} = 64. \end{aligned}$$

(d) $y = (1-x)^{1/3}, \quad -7 \le x \le 2.$



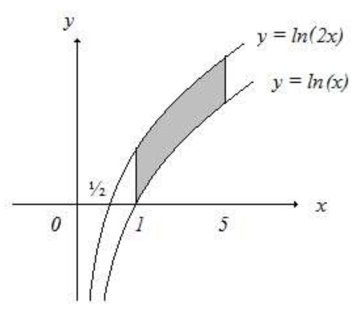
$$\begin{aligned} A &= \int_{-7}^{2} \left|(1-x)^{1/3}\right| \, dx = \int_{-7}^{2} |1-x|^{1/3} \, dx = \int_{-7}^{1} (1-x)^{1/3} \, dx - \int_{1}^{2} (1-x)^{1/3} \, dx \\ &= \int_{0}^{8} u^{1/3} \, du - \int_{-1}^{0} u^{1/3} \, du = \frac{3}{4}\left[u^{4/3}\right]_0^8 - \frac{3}{4}\left[u^{4/3}\right]_{-1}^0 = \frac{51}{4}. \end{aligned}$$

(e) $y = x\sqrt{4-x^2}, \quad -2 \le x \le 2.$



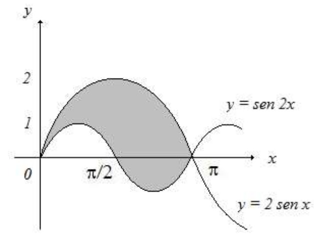
$$\begin{aligned} A &= \int_{-2}^{2} \left|x\sqrt{4-x^2}\right| \, dx = \int_{-2}^{2} |x| \sqrt{4-x^2} \, dx = -\int_{-2}^{0} x\sqrt{4-x^2} \, dx + \int_{0}^{2} x\sqrt{4-x^2} \, dx \\ &= \frac{1}{2}\int_{0}^{4} \sqrt{u} \, du + \frac{1}{2}\int_{0}^{4} \sqrt{u} \, du = \int_{0}^{4} \sqrt{u} \, du = \frac{2}{3}\left[u^{3/2}\right]_0^4 = \frac{16}{3}. \end{aligned}$$

2. (a) $y = \ln x$ y $y = \ln(2x), \quad 1 \le x \le 5.$



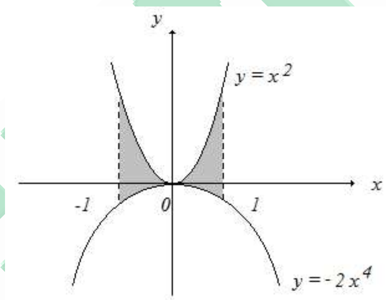
$$A = \int_{1}^{5} |\ln(2x) - \ln x| \, dx = \int_{1}^{5} (\ln 2 + \ln x - \ln x) \, dx = \int_{1}^{5} \ln 2 \, dx = (\ln 2)\,[x]_1^5 = 4\ln 2.$$

(b) $y = 2\,\mathrm{sen}\, x$ y $y = \mathrm{sen}\,(2x)\,, \quad 0 \leq x \leq \pi.$



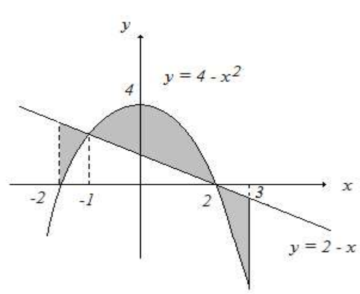
$$
\begin{aligned}
A &= \int_0^{\pi} |2\,\mathrm{sen}\, x - \mathrm{sen}\,(2x)| \; dx = \int_0^{\pi} [2\,\mathrm{sen}\, x - \mathrm{sen}\,(2x)] \; dx \\
&= 2 \int_0^{\pi} \mathrm{sen}\, x \; dx - \int_0^{\pi} \mathrm{sen}\,(2x) \; dx = 2 \int_0^{\pi} \mathrm{sen}\, x \; dx - \frac{1}{2} \int_0^{2\pi} \mathrm{sen}\, u \; du \\
&= -2\,[\cos x]_0^{\pi} + \frac{1}{2}\,[\cos u]_0^{2\pi} = -2(\underbrace{\cos \pi}_{-1} - \underbrace{\cos 0}_{1}) + \frac{1}{2}(\underbrace{\cos\,(2\pi)}_{1} - \underbrace{\cos 0}_{1}) = 4.
\end{aligned}
$$

(c) $y = x^2$ y $y = -2x^4, \quad -1 \leq x \leq 1.$



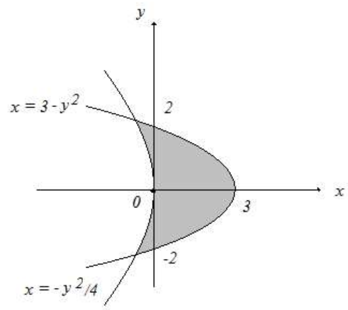
$$
\begin{aligned}
A &= \int_{-1}^{1} \left| x^2 - \left(-2x^4\right) \right| \; dx = \int_{-1}^{1} \left| x^2 + 2x^4 \right| \; dx = \int_{-1}^{1} \left[x^2 + 2x^4 \right] \; dx \\
&= 2 \int_0^{1} \left[x^2 + 2x^4 \right] \; dx = 2 \left[\frac{x^3}{3} + \frac{2x^5}{5} \right]_0^1 = \frac{22}{15}.
\end{aligned}
$$

(d) $y = 4 - x^2$ y $y = 2 - x, \quad -2 \leq x \leq 3.$



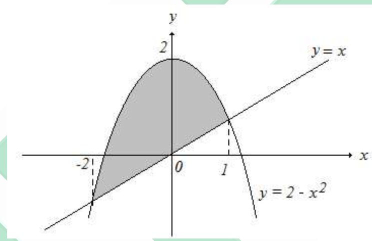
$$
\begin{aligned}
A &= \int_{-2}^{3} \left|(2-x)-\left(4-x^2\right)\right| \, dx = \int_{-2}^{3} \left|x^2-x-2\right| \, dx \\
&= \int_{-2}^{-1} \left(x^2-x-2\right) \, dx + \int_{-1}^{2} \left(-x^2+x+2\right) \, dx + \int_{2}^{3} \left(x^2-x-2\right) \, dx \\
&= \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-2}^{-1} + \left[-\frac{x^3}{3}+\frac{x^2}{2}+2x\right]_{-1}^{2} + \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{2}^{3} = \frac{49}{6}.
\end{aligned}
$$

(e) $x+y^2=3$ y $4x+y^2=0, \quad -2 \le y \le 2.$



$$A = \int_{-2}^{2} \left|(3-y^2)-\left(-\frac{y^2}{4}\right)\right| \, dy = \int_{-2}^{2} \left|3-\frac{3y^2}{4}\right| \, dy = \int_{-2}^{2} \left[3-\frac{3y^2}{4}\right] \, dy = \left[3y-\frac{y^3}{4}\right]_{-2}^{2} = 8.$$

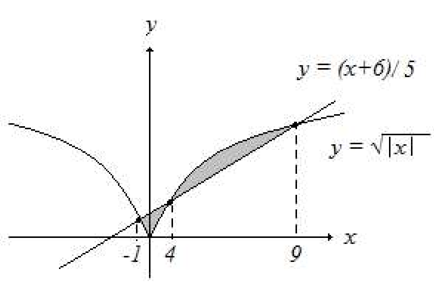
3. (a) $y=2-x^2$ y $y=x$.



$$
\begin{gathered}
2-x^2=x \\
x^2+x-2=0 \\
(x+2)(x-1)=0 \\
\therefore \; x=-2 \;\text{ o }\; x=1.
\end{gathered}
$$

$$A = \int_{-2}^{1} \left[2-x^2-x\right] \, dx = \left[2x-\frac{x^3}{3}-\frac{x^2}{2}\right]_{-2}^{1} = \frac{9}{2}.$$

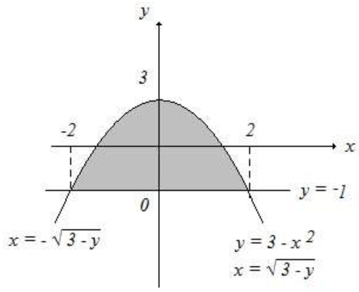
(b) $y=\sqrt{|x|}$ y $5y=x+6$.



$$
\begin{gathered}
\sqrt{|x|} = \tfrac{x+6}{5}, \quad x \ge -6 \\
25\,|x| = (x+6)^2 \\
\text{i) } x \ge 0 \Longrightarrow 25x = (x+6)^2 \\
x^2-13x+36=0 \\
(x-4)(x-9)=0 \\
\therefore \; x=4 \;\text{ o }\; x=9. \\
\text{ii) } -6 \le x < 0 \Longrightarrow -25x = (x+6)^2 \\
x^2+37x+36=0 \\
(x+1)(x+36)=0 \\
\therefore \; x=-1.
\end{gathered}
$$

$$A = \int_{-1}^{0}\left[\left(\frac{x}{5}+\frac{6}{5}\right)-\sqrt{-x}\right]\,dx+\int_{0}^{4}\left[\left(\frac{x}{5}+\frac{6}{5}\right)-\sqrt{x}\right]\,dx$$
$$+\int_{4}^{9}\left[\sqrt{x}-\left(\frac{x}{5}+\frac{6}{5}\right)\right]\,dx=\frac{5}{3}.$$

4. $y = 3 - x^2$ y $y = -1$.



$$\begin{gathered} 3 - x^2 = -1 \\ x^2 = 4 \\ |x| = 2 \\ \therefore\ x = 2 \ \text{ o } \ x = -2. \end{gathered}$$

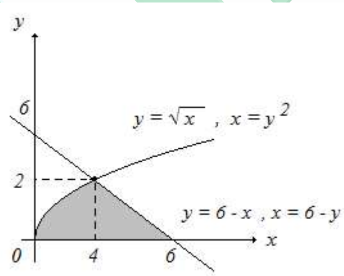
(a) Integrando con respecto a x (región y-simple):

$$A = \int_{-2}^{2}\left[\left(3-x^2\right)-(-1)\right]\,dx = \int_{-2}^{2}\left(4-x^2\right)\,dx = 2\int_{0}^{2}\left(4-x^2\right)\,dx = \frac{32}{3}.$$

(b) Integrando con respecto a y (región x-simple):

$$A = \int_{-1}^{3}\left[\sqrt{3-y}-\left(-\sqrt{3-y}\right)\right]\,dy = 2\int_{-1}^{3}\sqrt{3-y}\,dy = -2\int_{4}^{0}\sqrt{u}\,du$$
$$= 2\int_{0}^{4}\sqrt{u}\,du = \frac{4}{3}\left[u^{3/2}\right]_0^4 = \frac{32}{3}.$$

5. $y = \sqrt{x}$, $y = 6 - x$ y $y = 0$.



$$\begin{gathered} \sqrt{x} = 6 - x, \quad x \le 6 \\ x = 36 - 12x + x^2 \\ x^2 - 13x + 36 = 0 \\ (x-4)(x-9) = 0 \\ \therefore\ x = 4. \end{gathered}$$

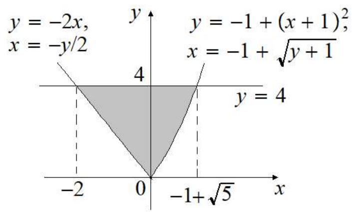
(a) Integrando con respecto a x (región y-simple):

$$A = \int_{0}^{4}\left(\sqrt{x}-0\right)\,dx + \int_{4}^{6}\left[(6-x)-0\right]\,dx = \frac{22}{3}.$$

(b) Integrando con respecto a y (región x-simple):

$$A = \int_{0}^{2}\left[(6-y)-y^2\right]\,dy = \frac{22}{3}.$$

6. $y=-2x$, $y=4$ y $y=-1+(x+1)^2$.



$$\begin{aligned} -1+(x+1)^2 &= 4\\ (x+1)^2 &= 5\\ |x+1| &= \sqrt{5}\\ x+1 &= \pm\sqrt{5}\\ \therefore\ x &= -1\pm\sqrt{5}. \end{aligned}$$

(a) Integrando con respecto a x (región y-simple):

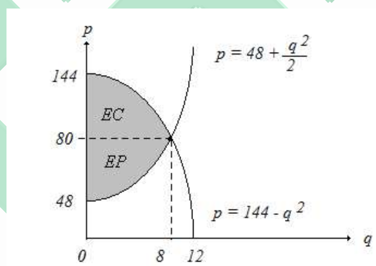
$$A = \int_{-2}^{0} [4-(-2x)]\ dx + \int_{0}^{-1+\sqrt{5}} \left[4-\left(-1+(x+1)^2\right)\right]\ dx = 6.787.$$

(b) Integrando con respecto a y (región x-simple):

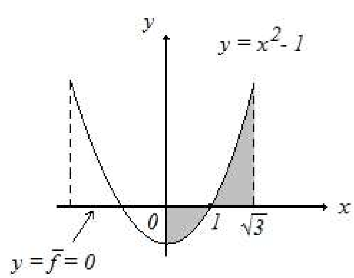
$$A=\int_0^4 \left[\left(-1+\sqrt{y+1}\right)-\left(-\frac{y}{2}\right)\right]\ dy = 6.787.$$

7. Los excedentes del consumidor EC y del productor EP están dados por

$$\begin{aligned} EC &= \int_0^8 \left[\left(144-q^2\right)-80\right]\ dq = \frac{1024}{3} \approx 341.3,\\ EP &= \int_0^8 \left[80-\left(48+\frac{q^2}{2}\right)\right]\ dq = \frac{512}{3} \approx 170.7 = \frac{EC}{2}. \end{aligned}$$



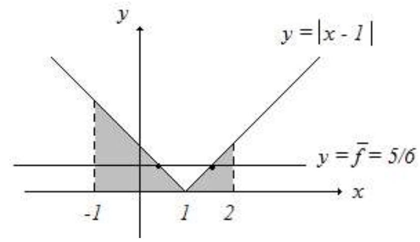
8. (a) $f(x)=x^2-1$ en $\left[0,\sqrt{3}\right]$.



$$\bar{f} = \frac{1}{\sqrt{3}-0}\int_0^{\sqrt{3}} \left(x^2-1\right)\ dx = \frac{1}{\sqrt{3}}\left[\frac{x^3}{3}-x\right]_0^{\sqrt{3}} = 0.$$

$$\therefore\quad \bar{f}=0\ \text{ y se alcanza en } x=1,\ \text{ ya que } f(1)=0.$$

(b) $f(x) = |x-1|$ en $[-1,2]$.



$$\bar{f} = \frac{1}{2-(-1)}\int_{-1}^{2} |x-1|\ dx = \frac{1}{3}\left[\int_{-1}^{1}(1-x)\ dx + \int_{1}^{2}(x-1)\ dx\right] = \frac{5}{6}.$$

$$\therefore \quad \bar{f} = \frac{5}{6} \text{ y se alcanza en } x_1 = \frac{1}{6},\ \text{y } x_2 = \frac{11}{6}, \text{ ya que } f\left(\frac{1}{6}\right) = f\left(\frac{11}{6}\right) = \frac{5}{6}.$$

9. Sea f continua y tal que $\int_1^2 f(x)\,dx = 4$. Como

$$\bar{f} = \frac{1}{2-1}\int_1^2 f(x)\ dx = \frac{1}{1}(4) = 4,$$

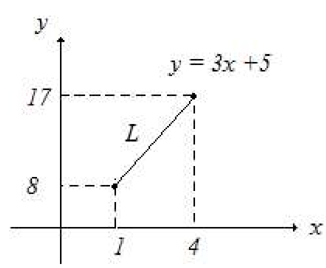
y como f es una función continua en $[1,2]$, por el Teorema del Valor Medio para integrales definidas existe $c \in [1,2]$ tal que $f(c) = 4$. Por lo tanto, $f(x) = 4$ al menos una vez en $[1,2]$.

10. La longitud L del segmento de la recta $y = 3x+5$ entre $x = 1$ y $x = 4$ está dada por

$$L = \int_1^4 \sqrt{1 + [y'(x)]^2}\ dx.$$

Como $y(x) = 3x+5$, por lo tanto $y'(x) = 3$, de modo que

$$L = \int_1^4 \sqrt{1+3^2}\ dx = \sqrt{10}\int_1^4 dx = 3\sqrt{10}.$$



En efecto, la distancia L entre el punto $(1,8)$ y el $(4,17)$ es

$$L = \sqrt{(4-1)^2 + (17-8)^2} = 3\sqrt{10}.$$

11. Como $y(x) = 2x^{3/2}$, por lo tanto $y'(x) = 3x^{1/2}$, de modo que

$$L = \int_{1/3}^{7} \sqrt{1 + [3x^{1/2}]^2}\ dx = \int_{1/3}^{7} \sqrt{1+9x}\ dx = \frac{1}{9}\int_4^{64} \sqrt{u}\ du = \frac{2}{27}\left[u^{3/2}\right]_4^{64} = \frac{112}{3}.$$

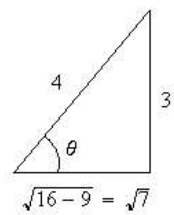
CÁLCULO III
TAREA 5 - SOLUCIONES
INTEGRALES RELACIONADAS CON LAS FUNCIONES TRIGONOMÉTRICAS INVERSAS
(Tema 1.6)

1. $\sec\left(\operatorname{sen}^{-1}\left(\frac{3}{4}\right)\right)$.

$$\text{Sea} \quad \theta = \operatorname{sen}^{-1}\left(\frac{3}{4}\right)$$

$$\therefore \quad \operatorname{sen}\theta = \frac{3}{4} = \frac{\text{cateto opuesto}}{\text{hipotenusa}}$$

$$\therefore \quad \sec\left(\operatorname{sen}^{-1}\left(\frac{3}{4}\right)\right) = \sec\theta = \frac{4}{\sqrt{7}}.$$



2.

$$\begin{aligned}
\log_5\left(2x + \tan^{-1}(y)\right) - \log_5 x &= 1 \\
\log_5\left(\frac{2x + \tan^{-1} y}{x}\right) &= 1 \\
5^{\log_5\left(\frac{2x+\tan^{-1} y}{x}\right)} &= 5^1 \\
\frac{2x + \tan^{-1} y}{x} &= 5 \\
2x + \tan^{-1} y &= 5x \\
\tan^{-1} y &= 3x \\
y &= \tan(3x).
\end{aligned}$$

3. (a) Si $y = \operatorname{sen}^{-1}\left(\sqrt{2}\,x\right)$, entonces

$$\frac{dy}{dx} = \frac{d\operatorname{sen}^{-1}\left(\sqrt{2}\,x\right)}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{2}\,x\right)^2}}\frac{d\left(\sqrt{2}\,x\right)}{dx} = \frac{\sqrt{2}}{\sqrt{1-2\,x^2}}.$$

(b) Si $y = \cot^{-1}(1/x) - \tan^{-1} x$, entonces

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d\left[\cot^{-1}(1/x) - \tan^{-1} x\right]}{dx} = -\frac{1}{1+(1/x)^2}\frac{d(1/x)}{dx} - \frac{1}{1+x^2} \\
&= -\frac{1}{1+(1/x)^2}\left(-\frac{1}{x^2}\right) - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0.
\end{aligned}$$

En otras palabras, $\frac{d\cot^{-1}(1/x)}{dx} = \frac{d\tan^{-1} x}{dx}$.

(c) Si $y = \ln(\tan^{-1} x)$, entonces

$$\frac{dy}{dx} = \frac{d\ln(\tan^{-1} x)}{dx} = \frac{1}{\tan^{-1} x}\frac{d\tan^{-1} x}{dx} = \frac{1}{\tan^{-1} x}\left(\frac{1}{1+x^2}\right) = \frac{1}{(1+x^2)\tan^{-1} x}.$$

(d) Si $y = e^{\tan^{-1}(x^5)}$, entonces

$$\frac{dy}{dx} = \frac{de^{\tan^{-1}(x^5)}}{dx} = e^{\tan^{-1}(x^5)}\frac{d\tan^{-1}(x^5)}{dx} = e^{\tan^{-1}(x^5)}\left(\frac{1}{1+(x^5)^2}\right)\frac{dx^5}{dx} = \frac{5x^4 e^{\tan^{-1}(x^5)}}{1+x^{10}}.$$

(e) Si $y = \csc^{-1}(e^x)$, entonces

$$\frac{dy}{dx} = \frac{d\csc^{-1}(e^x)}{dx} = -\frac{1}{|e^x|\sqrt{(e^x)^2-1}}\frac{de^x}{dx} = -\frac{e^x}{e^x\sqrt{e^{2x}-1}} = -\frac{1}{\sqrt{e^{2x}-1}}.$$

4. (a)

$$\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{1-(2x)^2}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\operatorname{sen}^{-1} u + C = \frac{1}{2}\operatorname{sen}^{-1}(2x) + C.$$

(b)

$$\begin{aligned}\int \frac{dy}{\sqrt{3+4y-4y^2}} &= \int \frac{dy}{\sqrt{4-(2y-1)^2}} = \frac{1}{2}\int \frac{2\,dy}{\sqrt{4-(2y-1)^2}} \\ &= \frac{1}{2}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{2}\operatorname{sen}^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{2}\operatorname{sen}^{-1}\left(\frac{2y-1}{2}\right) + C.\end{aligned}$$

(c)

$$\int \frac{e^t\,dt}{\sqrt{1-e^{2t}}} = \int \frac{e^t\,dt}{\sqrt{1-(e^t)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \operatorname{sen}^{-1} u + C = \operatorname{sen}^{-1}(e^t) + C.$$

(d) $\int \frac{dt}{7+3t^2}$

$$\begin{aligned}\int \frac{dt}{7+3t^2} &= \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}\,dt}{(\sqrt{7})^2+(\sqrt{3}t)^2} = \frac{1}{\sqrt{3}}\left[\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{\sqrt{3}t}{\sqrt{7}}\right)\right] + C \\ &= \frac{1}{\sqrt{21}}\tan^{-1}\left(\sqrt{\frac{3}{7}}t\right) + C.\end{aligned}$$

(e)

$$\int \frac{\operatorname{sen}\theta\,d\theta}{1+\cos^2\theta} = -\int \frac{du}{1+u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos\theta) + C.$$

(f)

$$\begin{aligned}\int \frac{3^x\,dx}{1+3^{2x}} &= \int \frac{3^x\,dx}{1+(3^x)^2} = \frac{1}{\ln 3}\int \frac{3^x\,\ln 3\,dx}{1+(3^x)^2} = \frac{1}{\ln 3}\int \frac{du}{1+u^2} \\ &= \frac{1}{\ln 3}\tan^{-1} u + C = \frac{1}{\ln 3}\tan^{-1}(3^x) + C.\end{aligned}$$

(g)

$$\begin{aligned}\int \frac{dx}{x\sqrt{5x^2-2}} &= \int \frac{\sqrt{5}\,dx}{\sqrt{5}x\sqrt{\left(\sqrt{5}x\right)^2-2}} = \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{\sqrt{5}x}{\sqrt{2}}\right| + C \\ &= \frac{1}{\sqrt{2}}\sec^{-1}\left|\sqrt{\frac{5}{2}}x\right| + C.\end{aligned}$$

(h)

$$\int \frac{dx}{(1+x^2)\tan^{-1} x} = \int \left(\frac{1}{\tan^{-1} x}\right)\frac{dx}{1+x^2} = \int \frac{du}{u} = \ln|u| + C = \ln\left|\tan^{-1} x\right| + C.$$

(i)

$$\int \frac{e^{\operatorname{sen}^{-1} x}\,dx}{\sqrt{1-x^2}} = \int e^{\operatorname{sen}^{-1} x}\frac{dx}{\sqrt{1-x^2}} = \int e^u\,du = e^u + C = e^{\operatorname{sen}^{-1} x} + C.$$

5. (a)

$$\begin{aligned}\int_0^1 \frac{4\,dx}{\sqrt{4-x^2}} &= 4\int_0^1 \frac{dx}{\sqrt{4-x^2}} = 4\left[\operatorname{sen}^{-1}\left(\frac{x}{2}\right)\right]_0^1 = 4\left[\operatorname{sen}^{-1}\left(\frac{1}{2}\right) - \operatorname{sen}^{-1}(0)\right] \\ &= 4\left[\frac{\pi}{6} - 0\right] = \frac{2\pi}{3}.\end{aligned}$$

(b)

$$\begin{aligned}\int_1^e \frac{dt}{t\left(1+\ln^2 t\right)} &= \int_1^e \frac{1}{\left(1+(\ln t)^2\right)}\frac{dt}{t} = \int_0^1 \frac{du}{1+u^2} = \left[\tan^{-1} u\right]_0^1 \\ &= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.\end{aligned}$$

(c)

$$\begin{aligned}\int_1^2 \frac{8\,dt}{t^2-2t+2} &= 8\int_1^2 \frac{dt}{(t-1)^2+1} = 8\int_0^1 \frac{du}{u^2+1} = 8\left[\tan^{-1} u\right]_0^1 \\ &= 8\left[\tan^{-1}(1) - \tan^{-1}(0)\right] = 8\left[\frac{\pi}{4} - 0\right] = 2\pi.\end{aligned}$$

(d) Método1:

$$\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1} x)\, dx}{x\sqrt{x^2-1}} = \int_{\sqrt{2}}^{2} \sec^2(\sec^{-1} x)\, \frac{dx}{x\sqrt{x^2-1}} = \int_{\sec^{-1}(\sqrt{2})}^{\sec^{-1}(2)} \sec^2 u\, du$$

$$= \int_{\pi/4}^{\pi/3} \sec^2 u\, du = [\tan u]_{\pi/4}^{\pi/3} = \tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right) = \sqrt{3} - 1.$$

Método 2:
Como $\sec(\sec^{-1} x) = x$, por lo tanto $\sec^2(\sec^{-1} x) = [\sec(\sec^{-1} x)]^2 = x^2$. De esta manera,

$$\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1} x)\, dx}{x\sqrt{x^2-1}} = \int_{\sqrt{2}}^{2} \frac{x^2\, dx}{x\sqrt{x^2-1}} = \int_{\sqrt{2}}^{2} \frac{x\, dx}{\sqrt{x^2-1}} = \int_{\sqrt{2}}^{2} \frac{2x\, dx}{2\sqrt{x^2-1}}$$

$$= \int_{1}^{3} \frac{du}{2\sqrt{u}} = \left[\sqrt{u}\right]_1^3 = \sqrt{3} - 1.$$

6. Las integrales indefinidas

$$\int \frac{dx}{\sqrt{1-x^2}} = \operatorname{sen}^{-1} x + C \quad \text{y} \quad \int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$$

son equivalentes. En efecto, si en la segunda expresión utilizamos la identidad

$$\operatorname{sen}^{-1} x + \cos^{-1} x = \frac{\pi}{2},$$

obtenemos

$$\int \frac{dx}{\sqrt{1-x^2}} = -\int \frac{-dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C = \operatorname{sen}^{-1} x - \frac{\pi}{2} + C.$$

Observamos que las dos integrales indefinidas difieren sólo en una constante, de modo que ambas respuestas son correctas.

CÁLCULO III
TAREA 6 - SOLUCIONES
TÉCNICAS DE INTEGRACIÓN
(Tema 1.7)

1. (a) $\displaystyle\int \frac{e^{2x}\,dx}{4+9e^{2x}}$.

Sea $u = 4+9e^{2x}$. Se tiene

$$u = 4+9e^{2x} \quad \Longrightarrow \quad du = 18e^{2x}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{e^{2x}\,dx}{4+9e^{2x}} &= \frac{1}{18}\int \frac{18e^{2x}\,dx}{4+9e^{2x}} = \frac{1}{18}\int \frac{du}{u} = \frac{1}{18}\ln|u| + C = \frac{1}{18}\ln\left|4+9e^{2x}\right| + C \\ &= \frac{1}{18}\ln\left(4+9e^{2x}\right) + C.\end{aligned}$$

(b) $\displaystyle\int \frac{e^{x}\,dx}{4+9e^{2x}}$.

Nota que

$$\int \frac{e^{x}\,dx}{4+9e^{2x}} = \int \frac{e^{x}\,dx}{4+(3e^{x})^2}.$$

Sea $u = 3e^{x}$. Se tiene

$$u = 3e^{x} \quad \Longrightarrow \quad du = 3e^{x}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{e^{x}\,dx}{4+9e^{2x}} &= \frac{1}{3}\int \frac{3e^{x}\,dx}{4+(3e^{x})^2} = \frac{1}{3}\int \frac{du}{4+u^2} = \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right] + C \\ &= \frac{1}{6}\tan^{-1}\left(\frac{3e^{x}}{2}\right) + C.\end{aligned}$$

(c) $\displaystyle\int_2^{16} \frac{1}{2x\sqrt{\ln x}}\,dx$.

Sea $u = \ln x$. Se tiene

$$u = \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\,dx,\; u(2) = \ln 2,\; u(16) = \ln 16.$$

Por lo tanto,

$$\begin{aligned}\int_2^{16} \frac{1}{2x\sqrt{\ln x}}\,dx &= \int_2^{16} \frac{1}{\sqrt{\ln x}}\left(\frac{1}{2x}\right)dx = \int_{\ln 2}^{\ln 16} \frac{du}{2\sqrt{u}} = \left[\sqrt{u}\right]_{\ln 2}^{\ln 16} \\ &= \sqrt{\ln 16} - \sqrt{\ln 2} = \sqrt{4\ln 2} - \sqrt{\ln 2} = \sqrt{\ln 2}.\end{aligned}$$

(d) $\displaystyle\int \frac{dx}{x\sqrt{1-\ln x}}$.

Sea $u = 1 - \ln x$. Se tiene

$$u = 1 - \ln x \quad \Longrightarrow \quad du = -\frac{1}{x}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{dx}{x\sqrt{1-\ln x}} &= -\int \frac{1}{\sqrt{1-\ln x}}\left(-\frac{1}{x}\right)dx = -\int \frac{du}{\sqrt{u}} = -2\sqrt{u} + C \\ &= -2\sqrt{1-\ln x} + C.\end{aligned}$$

(e) $\displaystyle\int \frac{dx}{x\sqrt{1-\ln^2 x}}$.

Nota que

$$\int \frac{dx}{x\sqrt{1-\ln^2 x}} = \int \frac{1}{\sqrt{1-(\ln x)^2}}\,\frac{1}{x}\,dx.$$

Sea $u = \ln x$. Se tiene

$$u = \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{dx}{x\sqrt{1-\ln^2 x}} &= \int \frac{1}{\sqrt{1-(\ln x)^2}}\left(\frac{1}{x}\right)dx = \int \frac{du}{\sqrt{1-u^2}} = \operatorname{sen}^{-1}(u) + C \\ &= \operatorname{sen}^{-1}(\ln x) + C.\end{aligned}$$

(f) $\displaystyle\int \frac{\ln x}{x + 4x\ln^2 x}\,dx$.

Nota que

$$\int \frac{\ln x}{x+4x\ln^2 x}\,dx = \int \frac{\ln x}{x\left(1+4\ln^2 x\right)}\,dx = \int \frac{1}{1+4(\ln x)^2}\,\frac{\ln x}{x}\,dx.$$

Sea $u = 1 + 4(\ln x)^2$. Se tiene

$$u = 1 + 4(\ln x)^2 \quad \Longrightarrow \quad du = 8(\ln x)\,\frac{1}{x}\,dx = \frac{8\ln x}{x}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{\ln x}{x+4x\ln^2 x}\,dx &= \frac{1}{8}\int \frac{1}{1+4(\ln x)^2}\left(\frac{8\ln x}{x}\right)dx = \frac{1}{8}\int \frac{du}{u} = \frac{1}{8}\ln|u| + C \\ &= \frac{1}{8}\ln\left(1+4\ln^2 x\right) + C.\end{aligned}$$

(g) $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx.$

Sea $u = 1+\sqrt{x}$. Se tiene

$$u = 1+\sqrt{x} \quad\Longrightarrow\quad du = \frac{1}{2\sqrt{x}}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx &= 2\int \sqrt{1+\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx = 2\int \sqrt{u}\,du = \frac{4}{3}u^{3/2}+C\\ &= \frac{4}{3}\left(1+\sqrt{x}\right)^{3/2}+C.\end{aligned}$$

(h) $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx.$

Sea $u = 1+\sqrt{x}$, de modo que $\sqrt{x} = u-1$. Se tiene

$$u = 1+\sqrt{x} \quad\Longrightarrow\quad du = \frac{1}{2\sqrt{x}}\,dx \quad\Longrightarrow\quad dx = 2\sqrt{x}\,du = 2\,(u-1)\,du.$$

Por lo tanto,

$$\begin{aligned}\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx &= 2\int \frac{(u-1)\,(u-1)}{u}\,du = 2\int\left(\frac{u^2-2u+1}{u}\right)du\\ &= 2\int\left(u-2+\frac{1}{u}\right)du = 2\left[\frac{u^2}{2}-2u+\ln|u|\right]+C\\ &= \left(1+\sqrt{x}\right)^2-4\left(1+\sqrt{x}\right)+2\ln\left|1+\sqrt{x}\right|+C.\\ &= \left(1+\sqrt{x}\right)^2-4\left(1+\sqrt{x}\right)+2\ln\left(1+\sqrt{x}\right)+C.\end{aligned}$$

2. (a) $\displaystyle\int \frac{x}{x-1}\,dx.$

$$\int \frac{x}{x-1}\,dx = \int \frac{x-1+1}{x-1}\,dx = \int\left(1+\frac{1}{x-1}\right)dx = x+\ln|x-1|+C.$$

(b) $\displaystyle\int \frac{\sqrt{x}}{x+1}\,dx.$

Sea $u = \sqrt{x}$. Se tiene

$$u = \sqrt{x} \quad\Longrightarrow\quad u^2 = x \quad\Longrightarrow\quad 2u\,du = dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{\sqrt{x}}{x+1}\,dx &= \int \frac{u}{u^2+1}\,2u\,du = 2\int\frac{u^2}{u^2+1}\,du = 2\int\frac{u^2+1-1}{u^2+1}\,du\\ &= 2\int\left(1-\frac{1}{u^2+1}\right)du = 2\left[u-\tan^{-1}u\right]+C\\ &= 2\sqrt{x}-2\tan^{-1}\left(\sqrt{x}\right)+C.\end{aligned}$$

(c) $\displaystyle\int \sqrt{\frac{x-1}{x^5}}\, dx.$

$$\int \sqrt{\frac{x-1}{x^5}}\, dx = \int \frac{1}{x^2}\sqrt{\frac{x-1}{x}}\, dx = \int \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\, dx.$$

Sea $u = 1 - \dfrac{1}{x}$. Se tiene

$$u = 1 - \frac{1}{x} \quad \Longrightarrow \quad du = \frac{1}{x^2}\, dx.$$

Por lo tanto,

$$\int \sqrt{\frac{x-1}{x^5}}\, dx = \int \sqrt{1-\frac{1}{x}}\left(\frac{1}{x^2}\right) dx = \int \sqrt{u}\, du = \frac{2}{3}u^{3/2}+C = \frac{2}{3}\left(1-\frac{1}{x}\right)^{3/2}+C.$$

(d) $\displaystyle\int (\sec x + \cot x)^2\ dx.$

$$\begin{aligned}
\int (\sec x + \cot x)^2\ dx &= \int \left(\sec^2 x + 2\sec x\cot x + \cot^2 x\right)\ dx \\
&= \int \left[\sec^2 x + 2\left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\operatorname{sen} x}\right) + \left(\csc^2 x - 1\right)\right] dx \\
&= \int \left(\sec^2 x + 2\csc x + \csc^2 x - 1\right)\ dx \\
&= \tan x - 2\ln|\csc x + \cot x| - \cot x - x + C.
\end{aligned}$$

(e) $\displaystyle\int \frac{1}{1+\operatorname{sen} x}\, dx.$

$$\begin{aligned}
\int \frac{1}{1+\operatorname{sen} x}\, dx &= \int \frac{1}{1+\operatorname{sen} x}\left(\frac{1-\operatorname{sen} x}{1-\operatorname{sen} x}\right) dx = \int \frac{1-\operatorname{sen} x}{1-\operatorname{sen}^2 x}\, dx = \int \frac{1-\operatorname{sen} x}{\cos^2 x}\, dx \\
&= \int \frac{1}{\cos^2 x}\, dx - \int \frac{\operatorname{sen} x}{\cos^2 x} d\ x = \int \sec^2 x\ dx - \int \sec x \tan x\ dx \\
&= \tan x - \sec x + C.
\end{aligned}$$

(f) $\displaystyle\int \frac{1-\operatorname{sen}\theta}{\cos\theta}\, d\theta.$

$$\begin{aligned}
\int \frac{1-\operatorname{sen}\theta}{\cos\theta}\, d\theta &= \int \frac{1-\operatorname{sen}\theta}{\cos\theta}\left(\frac{1+\operatorname{sen}\theta}{1+\operatorname{sen}\theta}\right) d\theta = \int \frac{1-\operatorname{sen}^2\theta}{\cos\theta\,(1+\operatorname{sen}\theta)}\, d\theta \\
&= \int \frac{\cos^2\theta}{\cos\theta\,(1+\operatorname{sen}\theta)}\, d\theta = \int \frac{\cos\theta}{1+\operatorname{sen}\theta}\, d\theta.
\end{aligned}$$

Sea $u = 1 + \operatorname{sen}\theta$. Se tiene

$$u = 1 + \operatorname{sen}\theta \quad \Longrightarrow \quad du = \cos\theta\ d\theta.$$

Por lo tanto,

$$\int \frac{1-\operatorname{sen}\theta}{\cos\theta}\, d\theta = \int \frac{\cos\theta}{1+\operatorname{sen}\theta}\, d\theta = \int \frac{du}{u} = \ln|u| + C = \ln|1+\operatorname{sen}\theta| + C.$$

(g) $\displaystyle\int \frac{1}{e^x+1}\,dx.$

$$\int \frac{1}{e^x+1}\,dx = \int \frac{1}{e^x+1}\left(\frac{e^{-x}}{e^{-x}}\right)dx = \int \frac{e^{-x}}{e^{-x}\left(e^x+1\right)}\,dx = \int \frac{e^{-x}}{1+e^{-x}}\,dx.$$

Sea $u = e^{-x}$. Se tiene

$$u = e^{-x} \quad \Longrightarrow \quad du = -e^{-x}\,dx.$$

Por lo tanto,

$$\begin{aligned}
\int \frac{1}{e^x+1}\,dx &= \int \frac{e^{-x}}{1+e^{-x}}\,dx = -\int \frac{\left(-e^{-x}\right)\,dx}{1+e^{-x}} = -\int \frac{du}{u} \\
&= -\ln|u| + C = -\ln\left|1+e^{-x}\right| + C = -\ln\left(1+e^{-x}\right) + C.
\end{aligned}$$

(h) $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx.$

$$\int \frac{1}{e^x+e^{-x}}\,dx = \int \frac{1}{e^x+e^{-x}}\left(\frac{e^x}{e^x}\right)dx = \int \frac{e^x}{e^x\left(e^x+e^{-x}\right)}\,dx = \int \frac{e^x}{e^{2x}+1}\,dx.$$

Sea $u = e^x$. Se tiene

$$u = e^x \quad \Longrightarrow \quad du = e^x\,dx.$$

Por lo tanto,

$$\begin{aligned}
\int \frac{1}{e^x+e^{-x}}\,dx &= \int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{e^x\,dx}{\left(e^x\right)^2+1} = \int \frac{du}{u^2+1} \\
&= \tan^{-1}(u) + C = \tan^{-1}\left(e^x\right) + C.
\end{aligned}$$

(i) $\displaystyle\int \frac{e^x}{e^x+e^{-x}}\,dx.$

$$\int \frac{e^x}{e^x+e^{-x}}\,dx = \int \frac{e^x}{e^x+e^{-x}}\left(\frac{e^x}{e^x}\right)dx = \int \frac{e^{2x}}{e^x\left(e^x+e^{-x}\right)}\,dx = \int \frac{e^{2x}}{e^{2x}+1}\,dx.$$

Sea $u = e^{2x}+1$. Se tiene

$$u = e^{2x}+1 \quad \Longrightarrow \quad du = 2e^{2x}\,dx.$$

Por lo tanto,

$$\begin{aligned}
\int \frac{e^x}{e^x+e^{-x}}\,dx &= \int \frac{e^{2x}}{e^{2x}+1}\,dx = \frac{1}{2}\int \frac{\left(2e^{2x}\right)\,dx}{e^{2x}+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C \\
&= \frac{1}{2}\ln\left|e^{2x}+1\right| + C = \frac{1}{2}\ln\left(e^{2x}+1\right) + C.
\end{aligned}$$

(j) $\displaystyle \frac{1}{2}\int \sqrt{e^x-1}\,dx.$

Sea $u=\sqrt{e^x-1}$. Se tiene

$$u=\sqrt{e^x-1} \quad\Longrightarrow\quad x=\ln(u^2+1) \quad\Longrightarrow\quad dx=\frac{2u}{u^2+1}\,du.$$

Por lo tanto,

$$\begin{aligned}\frac{1}{2}\int \sqrt{e^x-1}\,dx &= \frac{1}{2}\int u\,\frac{2u}{u^2+1}\,du=\int \frac{u^2}{u^2+1}\,du=\int \frac{u^2+1-1}{u^2+1}\,du\\ &= \int \left(1-\frac{1}{u^2+1}\right)du=u-\tan^{-1}(u)+C\\ &= \sqrt{e^x-1}-\tan^{-1}\left(\sqrt{e^x-1}\right)+C.\end{aligned}$$

(k) $\displaystyle \int \frac{8}{t^2-2t+2}\,dt.$

Completando cuadrados, se obtiene

$$\int \frac{8}{t^2-2t+2}\,dt=8\int \frac{dt}{(t-1)^2+1}.$$

Sea $u=t-1$. Se tiene

$$u=t-1 \quad\Longrightarrow\quad du=dt.$$

Por lo tanto,

$$\int \frac{8}{t^2-2t+2}\,dt=8\int \frac{dt}{(t-1)^2+1}=8\int \frac{du}{u^2+1}=8\tan^{-1}(u)+C=8\tan^{-1}(t-1)+C.$$

3. (a) $\displaystyle \int x\,\mathrm{sen}\,(3x)\,dx.$

Proponemos

$$\begin{aligned} u &= x \quad\Longrightarrow\quad du=dx\\ dv &= \mathrm{sen}\,(3x)\,dx \quad\Longrightarrow\quad v=-\frac{1}{3}\cos(3x).\end{aligned}$$

Por lo tanto,

$$\begin{aligned}\int x\,\mathrm{sen}\,(3x)\,dx &= -\frac{x}{3}\cos(3x)-\int\left(-\frac{1}{3}\cos(3x)\right)dx\\ &= -\frac{x}{3}\cos(3x)+\frac{1}{3}\int \cos(3x)\,dx\\ &= -\frac{x}{3}\cos(3x)+\frac{1}{9}\mathrm{sen}\,(3x)+C.\end{aligned}$$

(b) $\int_{0}^{2} xe^{-x/2}\, dx.$

Primero conviene determinar $\int xe^{-x/2}\, dx$. Para ello, proponemos

$$\begin{aligned} u &= x &\implies& \quad du = dx \\ dv &= e^{-x/2}\, dx &\implies& \quad v = -2e^{-x/2}. \end{aligned}$$

De esta manera,

$$\begin{aligned} \int xe^{-x/2}\, dx &= -2xe^{-x/2} - \int \left(-2e^{-x/2}\right) dx = -2xe^{-x/2} + 2\int e^{-x/2}\, dx \\ &= -2xe^{-x/2} + 2\left(-2e^{-x/2}\right) + C = -2xe^{-x/2} - 4e^{-x/2} + C \\ &= -2(x+2)e^{-x/2} + C. \end{aligned}$$

Por lo tanto,

$$\int_{0}^{2} xe^{-x/2}\, dx = -2\left[(x+2)e^{-x/2}\right]_0^2 = -2\left[4e^{-1} - 2e^{0}\right] = -8e^{-1} + 4.$$

(c) $\int x^2 e^{-x/2}\, dx.$

Proponemos

$$\begin{aligned} u &= x^2 &\implies& \quad du = 2x\, dx \\ dv &= e^{-x/2}\, dx &\implies& \quad v = -2e^{-x/2}. \end{aligned}$$

Por lo tanto,

$$\int x^2 e^{-x/2}\, dx = -2x^2 e^{-x/2} - \int \left(-2e^{-x/2}\right)(2x)\, dx = -2x^2 e^{-x/2} + 4\int xe^{-x/2}\, dx.$$

Ahora integramos por partes la nueva integral. Para ello, proponemos

$$\begin{aligned} u &= x &\implies& \quad du = dx \\ dv &= e^{-x/2}\, dx &\implies& \quad v = -2e^{-x/2}. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int x^2 e^{-x/2}\, dx &= -2x^2 e^{-x/2} + 4\int xe^{-x/2}\, dx \\ &= -2x^2 e^{-x/2} + 4\left[-2xe^{-x/2} - \int \left(-2e^{-x/2}\right) dx\right] \\ &= -2x^2 e^{-x/2} - 8xe^{-x/2} + 8\int e^{-x/2} dx \\ &= -2x^2 e^{-x/2} - 8xe^{-x/2} + 8\left(-2e^{-x/2}\right) + C \\ &= -2x^2 e^{-x/2} - 8xe^{-x/2} - 16e^{-x/2} + C \\ &= -2\left(x^2 + 4x + 8\right) e^{-x/2} + C. \end{aligned}$$

(d) $\int\limits_1^e \ln\sqrt{x}\,dx$.

Nota que
$$\int\limits_1^e \ln\sqrt{x}\,dx = \int\limits_1^e \ln x^{1/2}\,dx = \frac{1}{2}\int\limits_1^e \ln x\,dx.$$

Primero determinemos $\int \ln x\,dx$. Para ello, proponemos

$$\begin{aligned} u &= \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\,dx \\ dv &= dx \quad \Longrightarrow \quad v = x. \end{aligned}$$

De esta manera,

$$\int \ln x\,dx = x\ln x - \int (x)\,\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C.$$

Por lo tanto,

$$\begin{aligned} \int\limits_1^e \ln\sqrt{x}\,dx &= \frac{1}{2}\int\limits_1^e \ln x\,dx = \frac{1}{2}\left[x\ln x - x\right]_1^e = \frac{1}{2}\left[(e\ln e - e) - (\ln 1 - 1)\right] \\ &= \frac{1}{2}\left[(e-e)+1\right] = \frac{1}{2}. \end{aligned}$$

(e) $\int\limits_1^e x^2\ln x\,dx$.

Primero determinemos $\int x^2\ln x\,dx$. Para ello, proponemos

$$\begin{aligned} u &= \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\,dx \\ dv &= x^2\,dx \quad \Longrightarrow \quad v = \frac{x^3}{3}. \end{aligned}$$

De esta manera,

$$\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \int \left(\frac{x^3}{3}\right)\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.$$

Por lo tanto,

$$\begin{aligned} \int\limits_1^e x^2\ln x\,dx &= \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_1^e = \left(\frac{e^3}{3}\ln e - \frac{e^3}{9}\right) - \left(\frac{1}{3}\ln 1 - \frac{1}{9}\right) \\ &= \left(\frac{e^3}{3} - \frac{e^3}{9}\right) + \frac{1}{9} = \frac{2}{9}e^3 + \frac{1}{9}. \end{aligned}$$

(f) $\displaystyle\int \cos x \ \ln(\text{sen}x) \ dx.$

Proponemos

$$\begin{array}{rcl} u & = & \ln(\text{sen}\, x) \quad \Longrightarrow \quad du = \dfrac{\cos x}{\text{sen}\, x}\, dx \\ dv & = & \cos x \, dx \quad \Longrightarrow \quad v = \text{sen}\, x. \end{array}$$

Por lo tanto,

$$\begin{array}{rcl} \displaystyle\int \cos x \ \ln(\text{sen}x) \ dx & = & \text{sen}\, x \ \ln(\text{sen}\, x) - \displaystyle\int (\text{sen}\, x) \left(\dfrac{\cos x}{\text{sen}\, x}\right) dx \\ & = & \text{sen}\, x \ \ln(\text{sen}\, x) - \displaystyle\int \cos x \ dx \\ & = & \text{sen}\, x \ \ln(\text{sen}\, x) - \text{sen}\, x + C. \end{array}$$

(g) $\displaystyle\int \frac{e^{2x} \ dx}{\sqrt{1-e^x}}.$

Como

$$\int \frac{e^{2x} \ dx}{\sqrt{1-e^x}} = \int e^x \frac{e^x}{\sqrt{1-e^x}} \ dx,$$

proponemos

$$\begin{array}{rcl} u & = & e^x \quad \Longrightarrow \quad du = e^x \ dx \\ dv & = & \dfrac{e^x}{\sqrt{1-e^x}} \ dx \quad \Longrightarrow \quad v = -2\sqrt{1-e^x}. \end{array}$$

Por lo tanto,

$$\begin{array}{rcl} \displaystyle\int \frac{e^{2x} \ dx}{\sqrt{1-e^x}} & = & -2e^x\sqrt{1-e^x} - \displaystyle\int \left(-2\sqrt{1-e^x}\right)(e^x)\, dx \\ & = & -2e^x\sqrt{1-e^x} + 2\displaystyle\int e^x\sqrt{1-e^x} dx \\ & = & -2e^x\sqrt{1-e^x} - \dfrac{4}{3}(1-e^x)^{3/2} + C. \end{array}$$

(h) $\displaystyle\int \tan^{-1}(x) \ dx.$

Proponemos

$$\begin{array}{rcl} u & = & \tan^{-1}(x) \quad \Longrightarrow \quad du = \dfrac{dx}{1+x^2} \\ dv & = & dx \quad \Longrightarrow \quad v = x. \end{array}$$

Por lo tanto,

$$\int \tan^{-1}(x) \ dx = x\tan^{-1}(x) - \int \frac{x}{1+x^2} dx = x\tan^{-1}(x) - \frac{1}{2}\ln\left(1+x^2\right) + C.$$

(i) $\displaystyle\int e^{-x}\cos x\;dx.$

Proponemos

$$\begin{array}{rcl} u & = & e^{-x} \quad\Longrightarrow\quad du=-e^{-x}\;dx \\ dv & = & \cos x\;dx \quad\Longrightarrow\quad v=\operatorname{sen}x. \end{array}$$

Por lo tanto,

$$\begin{array}{rcl} \displaystyle\int e^{-x}\cos x\;dx & = & e^{-x}\operatorname{sen}x-\displaystyle\int(\operatorname{sen}x)\left(-e^{-x}\right)dx \\ & = & e^{-x}\operatorname{sen}x+\displaystyle\int e^{-x}\operatorname{sen}x\;dx. \end{array}$$

Ahora integramos por partes la nueva integral. Para ello, proponemos

$$\begin{array}{rcl} u & = & e^{-x} \quad\Longrightarrow\quad du=-e^{-x}\;dx \\ dv & = & \operatorname{sen}x\;dx \quad\Longrightarrow\quad v=-\cos x. \end{array}$$

Por lo tanto,

$$\begin{array}{rcl} \displaystyle\int e^{-x}\cos x\;dx & = & e^{-x}\operatorname{sen}x+\displaystyle\int e^{-x}\operatorname{sen}x\;dx \\ & = & e^{-x}\operatorname{sen}x+\left[-e^{-x}\cos x-\displaystyle\int(-\cos x)\left(-e^{-x}\right)dx\right] \\ & = & e^{-x}\operatorname{sen}x-e^{-x}\cos x-\displaystyle\int e^{-x}\cos x\;dx. \end{array}$$

Por lo tanto,

$$2\int e^{-x}\cos x\;dx=e^{-x}\left(\operatorname{sen}x-\cos x\right),$$

de modo que

$$\int e^{-x}\cos x\;dx=\frac{1}{2}e^{-x}\left(\operatorname{sen}x-\cos x\right)+C.$$

(j) $\displaystyle\int \operatorname{sen}(\ln x)\,dx.$

Proponemos

$$\begin{array}{rcl} u & = & \operatorname{sen}(\ln x) \quad\Longrightarrow\quad du=\cos(\ln x)\left(\dfrac{1}{x}\right)\;dx \\ dv & = & dx \quad\Longrightarrow\quad v=x. \end{array}$$

Por lo tanto,

$$\int \operatorname{sen}(\ln x)\,dx=x\operatorname{sen}(\ln x)-\int\cos(\ln x)\,dx.$$

Ahora integramos por partes la nueva integral. Para ello, proponemos

$$\begin{aligned} u &= \cos(\ln x) \quad \Longrightarrow \quad du = -\text{sen}(\ln x)\left(\frac{1}{x}\right)\ dx \\ dv &= dx \quad \Longrightarrow \quad v = x. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int \text{sen}(\ln x)\, dx &= x\text{sen}(\ln x) - \int \cos(\ln x)\, dx \\ &= x\text{sen}(\ln x) - \left[x\cos(\ln x) - \int (-\text{sen}(\ln x))\, dx\right] \\ &= x\left[\text{sen}(\ln x) - \cos(\ln x)\right] - \int \text{sen}(\ln x)\, dx. \end{aligned}$$

De esta manera,

$$2\int \text{sen}(\ln x)\, dx = x\left[\text{sen}(\ln x) - \cos(\ln x)\right],$$

de modo que

$$\int \text{sen}(\ln x)\, dx = \frac{x}{2}\left[\text{sen}(\ln x) - \cos(\ln x)\right] + C.$$

(k) $\displaystyle\int \cos(\sqrt{x})\ dx.$

Primero proponemos la sustitución $y = \sqrt{x}$. Se tiene

$$y = \sqrt{x} \quad \Longrightarrow \quad x = y^2 \quad \Longrightarrow \quad dx = 2y\ dy.$$

Por lo tanto,

$$\int \cos\left(\sqrt{x}\right)\ dx = 2\int y\cos y\ dy.$$

Esta última se integra por partes. Para ello, proponemos

$$\begin{aligned} u &= y \quad \Longrightarrow \quad du = dy \\ dv &= \cos y\ dy \quad \Longrightarrow \quad v = \text{sen}\, y. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int \cos\left(\sqrt{x}\right)\ dx &= 2\int y\cos y\ dy = 2\left[y\ \text{sen}\, y - \int \text{sen}\, y\ dy\right] \\ &= 2y\ \text{sen}\, y + 2\cos y + C = 2\sqrt{x}\ \text{sen}\ \left(\sqrt{x}\right) + 2\cos\left(\sqrt{x}\right) + C \end{aligned}$$

4. (a) $\displaystyle\int \frac{x+4}{x^2+5x+6}\,dx.$

Proponemos

$$\begin{aligned}\frac{x+4}{x^2+5x+6} &= \frac{x+4}{(x+2)(x+3)} = \frac{A}{x+2}+\frac{B}{x+3}\\ &= \frac{A(x+3)+B(x+2)}{(x+2)(x+3)} = \frac{(A+B)x+(3A+2B)}{(x+2)(x+3)},\end{aligned}$$

de donde

$$\begin{aligned}A+B &= 1\\ 3A+2B &= 4.\end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A=2 \text{ y } B=-1,$$

por lo que

$$\frac{x+4}{x^2+5x+6}=\frac{2}{x+2}-\frac{1}{x+3}.$$

Por lo tanto,

$$\begin{aligned}\int \frac{x+4}{x^2+5x+6}\,dx &= \int\left(\frac{2}{x+2}-\frac{1}{x+3}\right)dx = 2\int\frac{dx}{x+2}-\int\frac{dx}{x+3}\\ &= 2\ln|x+2|-\ln|x+3|+C.\end{aligned}$$

(b) $\displaystyle\int \frac{1}{(x+1)(x^2+1)}\,dx.$

Proponemos

$$\begin{aligned}\frac{1}{(x+1)(x^2+1)} &= \frac{A}{x+1}+\frac{Bx+C}{x^2+1}=\frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}\\ &= \frac{(A+B)x^2+(B+C)x+(A+C)}{(x+1)(x^2+1)},\end{aligned}$$

de donde

$$\begin{aligned}A+B &= 0\\ B+C &= 0\\ A+C &= 1.\end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A=C=1/2 \text{ y } B=-1/2,$$

por lo que

$$\frac{1}{(x+1)(x^2+1)}=\frac{1/2}{x+1}+\frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}.$$

Por lo tanto,

$$\begin{aligned}
\int \frac{1}{(x+1)(x^2+1)}\,dx &= \int \left(\frac{1/2}{x+1} + \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1} \right) dx \\
&= \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{dx}{x^2+1} \\
&= \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln\left|x^2+1\right| + \frac{1}{2}\tan^{-1}(x) + C.
\end{aligned}$$

(c) $\displaystyle\int \frac{x}{(x-1)^2}\,dx.$

Proponemos

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2} = \frac{Ax+(B-A)}{(x-1)^2},$$

de donde

$$\begin{aligned}
A &= 1 \\
B-A &= 0.
\end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A = B = 1,$$

por lo que

$$\frac{x}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2}.$$

Por lo tanto,

$$\begin{aligned}
\int \frac{x}{(x-1)^2}\,dx &= \int \left(\frac{1}{x-1} + \frac{1}{(x-1)^2} \right) dx = \int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} \\
& \ln|x-1| - \frac{1}{x-1} + C.
\end{aligned}$$

(d) $\displaystyle\int \frac{1}{x^2-x^3}\,dx.$

Proponemos

$$\begin{aligned}
\frac{1}{x^2-x^3} &= \frac{1}{x^2(1-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x} = \frac{Ax(1-x)+B(1-x)+Cx^2}{x^2(1-x)} \\
&= \frac{(C-A)x^2+(A-B)x+B}{x^2(1-x)},
\end{aligned}$$

de donde

$$\begin{aligned}
C-A &= 0 \\
A-B &= 0 \\
B &= 1.
\end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A = B = C = 1,$$

por lo que

$$\frac{1}{x^2 - x^3} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{1-x}.$$

Por lo tanto,

$$\begin{aligned}
\int \frac{1}{x^2 - x^3}\, dx &= \int \left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{1-x} \right) dx \\
&= \int \frac{dx}{x} + \int \frac{dx}{x^2} + \int \frac{dx}{1-x} \\
&= \ln|x| - \frac{1}{x} - \ln|1-x| + C.
\end{aligned}$$

(e) $\displaystyle\int \frac{x^2}{x^2-1}\, dx.$

Primero reducimos la fracción impropia en el integrando, quedando

$$\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1}.$$

De este modo,

$$\int \frac{x^2}{x^2-1}\, dx = \int \left(1 + \frac{1}{x^2-1} \right) dx.$$

Proponemos

$$\begin{aligned}
\frac{1}{x^2-1} &= \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1) + B(x+1)}{(x+1)(x-1)} \\
&= \frac{(A+B)x + (B-A)}{(x+1)(x-1)},
\end{aligned}$$

de donde

$$\begin{aligned}
A + B &= 0 \\
B - A &= 1.
\end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A = -1/2 \quad \text{y} \quad B = 1/2,$$

de modo que

$$\frac{1}{x^2-1} = -\frac{1/2}{x+1} + \frac{1/2}{x-1}.$$

Por lo tanto,

$$\begin{aligned}
\int \frac{x^2}{x^2-1}\, dx &= \int \left(1 + \frac{1}{x^2-1} \right) dx = \int \left(1 - \frac{1/2}{x+1} + \frac{1/2}{x-1} \right) dx \\
&= \int dx - \frac{1}{2} \int \frac{dx}{x+1} + \frac{1}{2} \int \frac{dx}{x-1} \\
&= x - \frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + C.
\end{aligned}$$

(f) $\displaystyle\int \frac{x^3 - x^2 + 1}{x^2 - x}\, dx.$

Primero reducimos la fracción impropia en el integrando, quedando

$$\frac{x^3 - x^2 + 1}{x^2 - x} = x + \frac{1}{x^2 - x}.$$

De este modo,

$$\int \frac{x^3 - x^2 + 1}{x^2 - x}\, dx = \int \left(x + \frac{1}{x^2 - x} \right) dx.$$

Proponemos

$$\begin{aligned} \frac{1}{x^2 - x} &= \frac{1}{x\,(x-1)} = \frac{A}{x} + \frac{B}{x-1} \\ &= \frac{A\,(x-1) + Bx}{x\,(x-1)} = \frac{(A+B)\,x - A}{x\,(x-1)}, \end{aligned}$$

de donde

$$\begin{aligned} A + B &= 0 \\ -A &= 1. \end{aligned}$$

Resolviendo el sistema de ecuaciones se obtiene

$$A = -1 \text{ y } B = 1,$$

de modo que

$$\frac{1}{x^2 - x} = -\frac{1}{x} + \frac{1}{x-1}.$$

Por lo tanto,

$$\begin{aligned} \int \frac{x^3 - x^2 + 1}{x^2 - x}\, dx &= \int \left(x + \frac{1}{x^2 - x} \right) dx = \int \left(x - \frac{1}{x} + \frac{1}{x-1} \right) dx \\ &= \int x\, dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} \\ &= \frac{x^2}{2} - \ln|x| + \ln|x-1| + C. \end{aligned}$$

5. (a) $\displaystyle\int \frac{\ln x}{x}\, dx.$

Método 1: Usando la sustitución $u = \ln x$.
Se tiene

$$u = \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\, dx.$$

Por lo tanto,

$$\int \frac{\ln x}{x}\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C = \frac{1}{2}\ln^2 x + C.$$

Método 2: Integrando por partes.
Proponemos

$$\begin{aligned} u &= \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\,dx \\ dv &= \frac{1}{x}\,dx \quad \Longrightarrow \quad v = \ln x. \end{aligned}$$

Por lo tanto,

$$\int \frac{\ln x}{x}\,dx = (\ln x)^2 - \int \frac{\ln x}{x}\,dx.$$

Así,

$$2\int \frac{\ln x}{x}\,dx = (\ln x)^2,$$

de donde

$$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2} + C.$$

(b) $\displaystyle\int \frac{\ln x}{x^2}\,dx.$

Se integra por partes. Para ello, proponemos

$$\begin{aligned} u &= \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\,dx \\ dv &= \frac{1}{x^2}\,dx \quad \Longrightarrow \quad v = -\frac{1}{x}. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int \frac{\ln x}{x^2}\,dx &= -\frac{1}{x}\ln x - \int \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx \\ &= -\frac{1}{x}\ln x - \frac{1}{x} + C. \end{aligned}$$

(c) $\displaystyle\int \frac{\operatorname{sen} x}{1+\cos^2 x}\,dx.$

Se propone la sustitución $u = \cos x$. Se tiene

$$u = \cos x \quad \Longrightarrow \quad du = -\operatorname{sen} x\,dx.$$

Por lo tanto,

$$\int \frac{\operatorname{sen} x}{1+\cos^2 x}\,dx = -\int \frac{du}{1+u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

(d) $\displaystyle\int \frac{1}{1+\cos x}\,dx.$

Se utilizan procedimientos algebraicos e identidades trigonométricas:

$$\begin{aligned} \int \frac{1}{1+\cos x}\,dx &= \int \frac{1}{1+\cos x}\left(\frac{1-\cos x}{1-\cos x}\right)\,dx = \int \frac{1-\cos x}{1-\cos^2 x}\,dx = \int \frac{1-\cos x}{\operatorname{sen}^2 x}\,dx \\ &= \int \left[\csc^2 x - \csc x \cot x\right]\,dx = -\cot x + \csc x + C. \end{aligned}$$

(e) $\displaystyle\int \ln\left(1+x^2\right)\,dx$.

Se integra por partes. Para ello, proponemos

$$\begin{aligned} u &= \ln\left(1+x^2\right) \quad \Longrightarrow \quad du = \frac{2x}{1+x^2}\,dx \\ dv &= dx \quad \Longrightarrow \quad v = x. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int \ln\left(1+x^2\right)\,dx &= x\ln\left(1+x^2\right) - 2\int \frac{x^2}{1+x^2}\,dx \\ &= x\ln\left(1+x^2\right) - 2\int \left(1-\frac{1}{1+x^2}\right)dx \\ &= x\ln\left(1+x^2\right) - 2x + 2\tan^{-1}(x) + C. \end{aligned}$$

(f) $\displaystyle\int \frac{x\,dx}{x^4+1}$.

Se integra por sustitución. Para ello, primero nota que

$$\int \frac{x\,dx}{x^4+1} = \int \frac{x\,dx}{\left(x^2\right)^2+1}.$$

Sea $u = x^2$. Se tiene

$$u = x^2 \quad \Longrightarrow \quad du = 2x\,dx.$$

Por lo tanto,

$$\int \frac{x\,dx}{x^4+1} = \frac{1}{2}\int \frac{2x\,dx}{\left(x^2\right)^2+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1}(u) + C = \frac{1}{2}\tan^{-1}\left(x^2\right) + C.$$

(g) $\displaystyle\int \frac{5x-3}{x^2-2x-3}\,dx$.

Se integra por fracciones parciales. Para ello, proponemos

$$\begin{aligned} \frac{5x-3}{x^2-2x-3} &= \frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} \\ &= \frac{A(x-3)+B(x+1)}{(x+1)(x-3)} = \frac{(A+B)x+(-3A+B)}{(x+1)(x-3)}, \end{aligned}$$

de donde

$$\begin{aligned} A+B &= 5 \\ -3A+B &= -3. \end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A = 2 \text{ y } B = 3,$$

por lo que

$$\frac{5x-3}{x^2-2x-3}=\frac{2}{x+1}+\frac{3}{x-3}.$$

Por lo tanto,

$$\begin{aligned}\int \frac{5x-3}{x^2-2x-3}\,dx &= \int\left(\frac{2}{x+1}+\frac{3}{x-3}\right)dx = 2\int\frac{dx}{x+1}+3\int\frac{dx}{x-3}\\ &= 2\ln|x+1|+3\ln|x-3|+C.\end{aligned}$$

(h) $\displaystyle\int \frac{dx}{x-\sqrt{x}}$

Se integra por sustitución. Para ello, nota que

$$\int\frac{dx}{x-\sqrt{x}}=\int\frac{dx}{\sqrt{x}\,(\sqrt{x}-1)}.$$

Sea $u=\sqrt{x}-1$. Se tiene

$$u=\sqrt{x}-1 \quad\Longrightarrow\quad du=\frac{1}{2\sqrt{x}}\,dx.$$

Por lo tanto,

$$\int\frac{dx}{x-\sqrt{x}}=2\int\frac{1}{\sqrt{x}-1}\left(\frac{1}{2\sqrt{x}}\right)dx=2\int\frac{du}{u}=2\ln|u|+C=2\ln\left|\sqrt{x}-1\right|+C.$$

(i) $\displaystyle\int \operatorname{sen}^{-1}(x)\;dx.$

Se integra por partes. Para ello, proponemos

$$\begin{aligned}u &= \operatorname{sen}^{-1}(x) \quad\Longrightarrow\quad du=\frac{dx}{\sqrt{1-x^2}}\\ dv &= dx \quad\Longrightarrow\quad v=x.\end{aligned}$$

Por lo tanto,

$$\begin{aligned}\int \operatorname{sen}^{-1}(x)\;dx &= x\operatorname{sen}^{-1}(x)-\int\frac{x}{\sqrt{1-x^2}}\,dx\\ &= x\operatorname{sen}^{-1}(x)+\sqrt{1-x^2}+C.\end{aligned}$$

(j) $\displaystyle\int\frac{x\,dx}{\sqrt{x+1}}.$

Método 1: Usando la sustitución $u=x+1$.
Se tiene

$$u=x+1 \quad\Longrightarrow\quad x=u-1 \quad\Longrightarrow\quad dx=du.$$

Por lo tanto,

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x+1}} &= \int \frac{u-1}{\sqrt{u}}\,du = \int \left(u^{1/2}-u^{-1/2}\right)du = \frac{2}{3}u^{3/2}-2u^{1/2}+C \\ &= \frac{2}{3}(x+1)^{3/2}-2(x+1)^{1/2}+C = \frac{2}{3}(x+1)^{1/2}(x-2)+C.\end{aligned}$$

Método 2: Usando la sustitución $u=\sqrt{x+1}$.
Se tiene

$$u=\sqrt{x+1} \quad\Longrightarrow\quad u^2=x+1 \quad\Longrightarrow\quad x=u^2-1 \quad\Longrightarrow\quad dx=2u\,du.$$

Por lo tanto,

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x+1}} &= \int \frac{u^2-1}{u}\,2u\,du = 2\int \left(u^2-1\right)du = \frac{2}{3}u^3-2u+C \\ &= \frac{2}{3}(x+1)^{3/2}-2(x+1)^{1/2}+C = \frac{2}{3}(x+1)^{1/2}(x-2)+C.\end{aligned}$$

Método 3: Integrando por partes.
Proponemos

$$\begin{aligned}u &= x \quad\Longrightarrow\quad du=dx \\ dv &= \frac{dx}{\sqrt{x+1}} \quad\Longrightarrow\quad v=2\sqrt{x+1}.\end{aligned}$$

Por lo tanto,

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x+1}} &= 2x\sqrt{x+1}-2\int \sqrt{x+1}dx = 2x\sqrt{x+1}-\frac{4}{3}(x+1)^{3/2}+C \\ &= \frac{2}{3}(x+1)^{1/2}(x-2)+C.\end{aligned}$$

(k) $\displaystyle\int \frac{dx}{\sqrt{x}+4x\sqrt{x}}$.

Se integra por sustitución. Para ello, nota que

$$\int \frac{dx}{\sqrt{x}+4x\sqrt{x}} = \int \frac{dx}{\sqrt{x}\,(1+4x)} = \int \frac{1}{1+\left(2\sqrt{x}\right)^2}\,\frac{1}{\sqrt{x}}\,dx.$$

Sea $u=2\sqrt{x}$. Se tiene

$$u=2\sqrt{x} \quad\Longrightarrow\quad du=\frac{1}{\sqrt{x}}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{dx}{\sqrt{x}+4x\sqrt{x}} &= \int \frac{1}{1+\left(2\sqrt{x}\right)^2}\left(\frac{1}{\sqrt{x}}\right)dx = \int \frac{1}{1+u^2}\,du = \tan^{-1}(u)+C \\ &= \tan^{-1}\left(2\sqrt{x}\right)+C.\end{aligned}$$

(l) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Se integra por procedimientos algebraicos, multiplicando el integrando por 1.

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{dx}{\sqrt{e^{2x}-1}}\left(\frac{e^x}{e^x}\right)dx = \int \frac{e^x\ dx}{e^x\sqrt{e^{2x}-1}}.$$

Sea $u = e^x$. Se tiene

$$u = e^x \quad \Longrightarrow \quad du = e^x\ dx.$$

Por lo tanto,

$$\begin{aligned}
\int \frac{dx}{\sqrt{e^{2x}-1}} &= \int \frac{e^x\ dx}{e^x\sqrt{e^{2x}-1}} = \int \frac{e^x\ dx}{e^x\sqrt{(e^x)^2-1}} = \int \frac{du}{u\sqrt{u^2-1}} \\
&= \sec^{-1}|u| + C = \sec^{-1}|e^x| + C = \sec^{-1}(e^x) + C.
\end{aligned}$$

(m) $\displaystyle\int \frac{dz}{\sqrt{3-2z-z^2}}$.

Se integra por procedimientos algebraicos, completando cuadrados.

$$\int \frac{dz}{\sqrt{3-2z-z^2}} = \int \frac{dz}{\sqrt{4-(z+1)^2}}.$$

Sea $u = z+1$. Se tiene

$$u = z+1 \quad \Longrightarrow \quad du = dz.$$

Por lo tanto,

$$\begin{aligned}
\int \frac{dz}{\sqrt{3-2z-z^2}} &= \int \frac{dz}{\sqrt{4-(z+1)^2}} = \int \frac{du}{\sqrt{4-u^2}} \\
&= \operatorname{sen}^{-1}\left(\frac{u}{2}\right) + C = \operatorname{sen}^{-1}\left(\frac{z+1}{2}\right) + C.
\end{aligned}$$

(n) $\displaystyle\int e^{\sqrt{x}}\ dx$.

Primero proponemos la sustitución $y = \sqrt{x}$. Se tiene

$$y = \sqrt{x} \quad \Longrightarrow \quad y^2 = x \quad \Longrightarrow \quad 2y\ dy = dx.$$

Por lo tanto,

$$\int e^{\sqrt{x}}\ dx = \int e^y\,(2y)\,dy = 2\int ye^y\ dy.$$

Esta última se integra por partes. Para ello, proponemos

$$\begin{aligned}
u &= y \quad \Longrightarrow \quad du = dy \\
dv &= e^y\ dy \quad \Longrightarrow \quad v = e^y.
\end{aligned}$$

Por lo tanto,

$$\begin{aligned}\int e^{\sqrt{x}}\,dx &= 2\int ye^{y}\,dy = 2\left[ye^{y}-\int e^{y}\,dy\right] = 2\left[ye^{y}-e^{y}\right]+C\\ &= 2\left(y-1\right)e^{y}+C = 2\left(\sqrt{x}-1\right)e^{\sqrt{x}}+C.\end{aligned}$$

(o) $\displaystyle\int \frac{x^4+x^2-1}{x^3+x}\,dx.$

Se integra por fracciones parciales. Primero reducimos la fracción impropia en el integrando, quedando

$$\frac{x^4+x^2-1}{x^3+x} = x-\frac{1}{x^3+x} = x-\frac{1}{x\left(x^2+1\right)}.$$

De este modo,

$$\int \frac{x^4+x^2-1}{x^3+x}\,dx = \int\left(x-\frac{1}{x\left(x^2+1\right)}\right)\,dx.$$

Proponemos

$$\begin{aligned}\frac{1}{x\left(x^2+1\right)} &= \frac{A}{x}+\frac{Bx+C}{x^2+1} = \frac{A\left(x^2+1\right)+\left(Bx+C\right)x}{x\left(x^2+1\right)}\\ &= \frac{\left(A+B\right)x^2+Cx+A}{x\left(x^2+1\right)},\end{aligned}$$

de donde

$$\begin{aligned}A+B &= 0\\ C &= 0\\ A &= 1.\end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A=1,\ B=-1 \text{ y } C=0,$$

por lo que

$$\frac{1}{x\left(x^2+1\right)} = \frac{1}{x}+\frac{-x}{x^2+1}.$$

Por lo tanto,

$$\begin{aligned}\int \frac{x^4+x^2-1}{x^3+x}\,dx &= \int\left(x-\frac{1}{x\left(x^2+1\right)}\right)\,dx = \int\left(x-\left(\frac{1}{x}+\frac{-x}{x^2+1}\right)\right)\,dx\\ &= \int xdx-\int\frac{dx}{x}+\int\frac{x}{x^2+1}dx\\ &= \frac{x^2}{2}-\ln|x|+\frac{1}{2}\ln\left(x^2+1\right)+C.\end{aligned}$$

(p) $\int x^3 e^{x^2}\, dx.$

Método 1: Usando la sustitución $z = x^2$.
Proponemos

$$z = x^2 \quad \Longrightarrow \quad x = \sqrt{z} \quad \Longrightarrow \quad dx = \frac{1}{2\sqrt{z}} dz.$$

Por lo tanto,

$$\int x^3 e^{x^2}\, dx = \int z^{3/2} e^z \left(\frac{1}{2\sqrt{z}}\right) dz = \frac{1}{2}\int z e^z\, dz.$$

Ahora integramos por partes la nueva integral. Para ello, proponemos

$$\begin{aligned} u &= z \quad \Longrightarrow \quad du = dz \\ dv &= e^z\, dz \quad \Longrightarrow \quad v = e^z. \end{aligned}$$

Por lo tanto,

$$\frac{1}{2}\int z e^z\, dz = \frac{1}{2}\left[z e^z - \int e^z\, dz\right] = \frac{1}{2}\left[z e^z - e^z\right] + C.$$

Así,

$$\int x^3 e^{x^2}\, dx = \frac{1}{2}\left[z - 1\right] e^z + C = \frac{1}{2}\left[x^2 - 1\right] e^{x^2} + C.$$

Método 2: Integrando por partes directamente.
Puede integrarse directamente por partes, pero de una manera cuidadosa. Como

$$\int x^3 e^{x^2}\, dx = \int x^2 \left(x e^{x^2}\right)\, dx,$$

proponemos

$$\begin{aligned} u &= x^2 \quad \Longrightarrow \quad du = 2x\, dx \\ dv &= x e^{x^2}\, dx \quad \Longrightarrow \quad v = \frac{1}{2} e^{x^2}. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int x^3 e^{x^2}\, dx &= \frac{1}{2} x^2 e^{x^2} - \int \left(\frac{1}{2} e^{x^2}\right)(2x)\, dx = \frac{1}{2} x^2 e^{x^2} - \int x e^{x^2} dx \\ &= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C = \frac{1}{2}\left(x^2 - 1\right) e^{x^2} + C. \end{aligned}$$